THE GENERALIZED SMALE CONJECTURE
FOR 3-MANIFOLDS WITH GENUS 2
ONE-SIDED HEEGAARD SPLITTINGS

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Abstract. The Generalized Smale Conjecture asserts that the inclusion of the group of isometries of any closed 3-manifold of constant positive curvature into the group of all its diffeomorphisms is a homotopy equivalence. In this paper, the conjecture is proven for all such 3-manifolds that contain a one-sided Klein bottle, except for the lens space $L(4,1)$.

Introduction

The Smale Conjecture, proven by Hatcher [18], asserts that if $M$ is the 3-sphere with the standard constant curvature metric, the inclusion $\text{Isom}(M) \to \text{Diff}(M)$ from the isometry group to the diffeomorphism group is a homotopy equivalence. The Generalized Smale Conjecture asserts this whenever $M$ is a closed 3-manifold with a metric of constant positive curvature. All known examples of closed 3-manifolds with finite fundamental group admit such metrics, whose isometry groups can be completely calculated [26]. Through work of many authors, including [3, 5, 6, 8, 31, 32, 33], it is known that $\text{Isom}(M) \to \text{Diff}(M)$ is bijective on the sets of path components (see [26]).

The Generalized Smale Conjecture is an instance of the general principle, first realized by Thurston, that 3-manifold topology is profoundly affected by the existence and behavior of geometric structures. In the positive curvature case, the Generalized Smale Conjecture suggests that not only the topological structure but also the group of all smooth automorphisms is controlled by the geometry. For (compact) 3-manifolds whose interiors have constant negative curvature and finite volume, the analogous expectation holds true: the composition $\text{Isom}(M) \to \text{Out}(\pi_1(M)) \to \text{Diff}(M)$ is a homotopy equivalence for finite-volume hyperbolic 3-manifolds [12] (for Haken hyperbolic 3-manifolds, this was already known by Mostow Rigidity, Waldhausen’s Theorem, and work of Hatcher discussed below). In contrast, when the manifold has interior of constant negative curvature and infinite volume,
or has constant zero curvature, a diffeomorphism will not in general be isotopic to an isometry (said differently, the diffeomorphism group may have more components than the isometry group). Even in these cases, however, Waldhausen’s Theorem and Hatcher’s work show that $\text{Isom}(M) \to \text{Diff}(M)$ is always a homotopy equivalence when one restricts to the connected components of the identity diffeomorphism.

The isometry groups of 3-manifolds with constant curvature are topologically rather simple: for the positive curvature cases they are compact Lie groups of dimension at most 6 [26], for the zero curvature cases they can be shown to be compact Lie groups of dimension at most 3, and in the case of negative curvature Mostow Rigidity implies that they are finite. Consequently, for the known cases of the Smale Conjecture, the homotopy type of $\text{Diff}(M)$ is rather uncomplicated. With the $C^r$-topologies with $1 \leq r < \infty$, diffeomorphism groups are Hilbert manifolds, while for the $C^{\infty}$-topology the local model is a Fréchet space (see for example section 1.2 of [4]). In either case, they are homeomorphic whenever they share the same homotopy type ([7] or [19] for the Hilbert manifold case, [1] for the Fréchet manifold case). Consequently, the hyperbolic case of [12] and the results we shall prove in this paper furnish many examples of 3-manifolds whose diffeomorphism groups are homeomorphic. This contrasts with the fact that the isomorphism type of $\text{Diff}(M)$ determines $M$ for all manifolds. That is, an abstract isomorphism between the diffeomorphism groups of two differentiable manifolds must be induced by a diffeomorphism between the manifolds [38, 9, 4].

For sufficiently large $\mathbb{P}^2$-irreducible 3-manifolds, Hatcher ([16], combined with [18]), extending earlier work of Laudenbach [25], proved that the components of $\text{Diff}(M \text{ rel } \partial M)$ have the expected homotopy types. The main part of the argument is to show that the space of imbeddings of a 2-sided incompressible surface $F$ that are disjoint from a parallel copy of $F$ is a deformation retract of the space of all imbeddings of $F$ that are isotopic to the inclusion relative to $\partial F$.

For manifolds that are not sufficiently large and therefore do not contain a 2-sided incompressible surface, one may try to use a 1-sided incompressible surface instead. If $M$ is orientable, irreducible, and not sufficiently large, and contains a 1-sided incompressible surface $K$, then by theorem 4 of [31], $M - K$ is an open handlebody. When the complement of a 1-sided surface $K$ in $M$ is an open handlebody, we say that $(M, K)$ is a 1-sided Heegaard splitting. The genus of the splitting is the (nonorientable) genus of $K$. In particular, when $K$ is a Klein bottle, $M - K$ is an open solid torus.

Let $K_0$ be a Klein bottle and write $\pi_1(K_0) = \langle a, b | bab^{-1} = a^{-1} \rangle$. Up to isotopy there are two possible Seifert fiberings of $K_0$ by circles. The “meridinal” fibering is nonsingular, and its fibers represent (the conjugacy class of) $a$. The “longitudinal” fibering has two exceptional fibers, which are the center circles of Möbius bands and whose nearby fibers wrap twice around the exceptional ones. Its generic fiber represents $b^2$. When $M$ has a genus-2 1-sided Heegaard splitting and $M$ is not $S^2 \times S^1$ or $\mathbb{RP}^3 \# \mathbb{RP}^3$, 

each of these fiberings extends to a Seifert fibering of $M$. (For $S^2 \times S^1$, the $a$ curve is contractible in the complementary torus, and for $\mathbb{RP}^3 \# \mathbb{RP}^3$ the $b^2$ curve is contractible.) There are four cases:

I. Those for which neither the meridinal nor the longitudinal fibering is nonsingular on the complement of $K_0$.

II. Those for which only the longitudinal fibering is nonsingular on the complement of $K_0$. These are the lens spaces $L(4n, 2n - 1)$, $n \geq 2$.

III. Those for which only the meridinal fibering is nonsingular on the complement of $K_0$. These have binary dihedral fundamental groups.

IV. The lens space $L(4, 1)$, for which both the meridinal and longitudinal fiberings are nonsingular on the complement of $K_0$.

In [21], N. Ivanov announced the Generalized Smale Conjecture for cases I and II, and in [22] he gave a detailed proof for case I. In the remaining sections of this paper, drawing upon methods of Ivanov and Hatcher, we will prove the Conjecture for all cases except case IV. More precise statements and an outline of the argument will be given in section 1, after fixing notation.

Ivanov’s announced results were used in [11] to construct examples of homeomorphisms of reducible 3-manifolds that are homotopic but not isotopic. Our results show that the construction applies to a larger class of 3-manifolds. In [30], our work was applied to the classification problem for 3-manifolds which have metrics of positive Ricci curvature and universal cover $S^3$.

The Generalized Smale Conjecture has attracted the interest of physicists studying the theory of quantum gravity. Certain physical configuration spaces can be realized as the quotient space of a principal Diff$_1(M, x_0)$-bundle with contractible total space, where Diff$_1(M, x_0)$ denotes the subspace of Diff($M, x_0$) that induce the identity on the tangent space to $M$ at $x_0$. (This group is homotopy equivalent to Diff($M \# D^3$ rel $\partial D^3$).) Consequently the loop space of the configuration space is weakly homotopy equivalent to Diff$_1(M, x_0)$. Physical significance of $\pi_0(\text{Diff}(M))$ for quantum gravity was first pointed out in [10]. See also [2], [14], [20], [37], [42]. The significance of some higher homotopy groups of Diff($M$) is examined in [13].

Throughout this paper, we assume that all spaces of diffeomorphisms and imbeddings have the $C^r$-topology for some $1 \leq r \leq \infty$. This comes into play in the following ways. First, with the $C^r$-topology, the space of diffeomorphisms of a manifold has the homotopy type of a CW-complex [29], so that homotopy equivalences of this space can be detected by their effect on homotopy groups. Second, we use the fact that the restriction of diffeomorphisms and imbeddings to submanifolds is a fibration, as was proven by Palais [28] and Cerf [8], and extended to spaces of fiber-preserving mappings in [24]. Finally, we use the elementary fact that the property of being transverse to a submanifold is an open condition in the $C^r$-topology.
An earlier version of this paper was circulated in the late 1980’s, and has been cited a number of times in the scientific literature. In this new version, the essential mathematical content is unchanged, but a considerable amount of detail has been added. Also, various “folk” theorems about fibrations of spaces of diffeomorphisms and imbeddings, heavily used in our arguments, have been put on firm ground by the work in [24].

1. Notation and statement of results

Let $K_0$ be a Klein bottle with $\pi_1(K_0) = \langle a, b \mid b^{-1}ab = a^{-1}\rangle$. The four homotopy classes of (unoriented) essential simple closed curves on $K_0$ are $b$, $ab$, $a$, and $b^2$, with $b$ and $ab$ orientation-reversing and $a$ and $b^2$ orientation-preserving.

Let $P$ be the orientable I-bundle over $K_0$. The free abelian group $\pi_1(\partial P)$ is generated by (elements homotopic in $P$ to) $a$ and $b^2$.

Let $R$ be a solid torus containing a meridional 2-disc with boundary $C$, a circle in $\partial R$. For a pair $(m, n)$ of relatively prime integers, the 3-manifold $M(m,n)$ is formed by identifying $\partial R$ and $\partial P$ in such a way that $C$ is attached along a simple closed curve representing the element $a^mb^{2n}$. If $m = 0$, the resulting manifold is $\mathbb{R}P^3 \# \mathbb{R}P^3$, while if $n = 0$ it is $S^2 \times S^1$, so from now on we assume that both $m$ and $n$ are nonzero. Since $M(-m,n) = M(m,n)$ and $M(-m,-n) = M(m,n)$, we can and always will assume that both $m$ and $n$ are positive.

Each fibering of $K_0$ produces a Seifert fibering of $M(m,n)$. If $K_0$ has the longitudinal fibering, then in $\partial P$ the fiber represents $b^2$. The meridian circle $C$ of $R$ equals $ma + nb^2$. Choosing $p$ and $q$ so that $mp - nq = 1$, the element $L = qa + pb^2$ is a longitude of $R$, since the intersection number $C \cdot L = mp - nq = 1$. We find that $b^2 = mL - qC$, so on $R$ the Seifert fibering has an exceptional fiber of order $m$. If instead $K_0$ has the meridional fibering, then the fiber represents $a$ in $\partial R$, and since $a = pC - nL$, the exceptional fiber in $R$ is of order $n$. This shows that in terms of $m$ and $n$, the cases discussed in the introduction are as follows: I is $m > 1$ and $n > 1$, II is $m = 1$ and $n > 1$, III is $m > 1$ and $n = 1$, and IV is $m = n = 1$.

The fundamental group of $M(m,n)$ has a presentation $\langle a, b \mid bab^{-1} = a^{-1}, a^mb^{2n} = 1 \rangle$. Note that $a^{2m} = 1$ and $b^{4n} = 1$.

If $n$ is odd, then $\pi_1(M(m,n)) \cong C_n \times D_{4m}^*$, where $C_n$ is cyclic and

$$D_{4m}^* = \langle x, y \mid x^2 = y^m = (xy)^2 \rangle$$

is the binary dihedral group. The $C_n$ factor is generated by $b^4$ and the $D_{4m}^*$ factor by $x = b^n$ and $y = a$.

If $n$ is even, write $C_{4n} = \langle t \mid t^{4n} = 1 \rangle$. Let $\Delta$ be the diagonal subgroup of index 2 in $C_{4n} \times D_{4m}^*$. That is, there is a unique homomorphism from $C_{4n}$ onto $C_2$, and, since $m$ is odd, a unique homomorphism from $D_{4m}^*$ onto $C_2$. The latter sends $y$ to 1. Combining these homomorphisms sends $C_{4n} \times D_{4m}^*$ onto $C_2$ with kernel $\Delta$. The element $(t^{2n}, y^n)$ is a central involution in $\Delta$, with
and \( \pi_1(M(m, n)) \) is isomorphic to \( \Delta/\langle\langle t^{2n}, y^m \rangle\rangle \). The correspondence is that \( a = (1, y) \) and \( b = (t, x) \) (so \( a^m = (1, x^2) \) and \( a^m b^2 = (t^2, 1) \)).

When \( m = 1 \), the groups reduce in both cases to a cyclic group of order \( 4n \). From [32], \( M(1, n) = L(4n, 2n - 1) \). This homeomorphism can also be seen directly as follows. Let \( T \) be a solid torus with \( H_1(\partial T) \) the free abelian group generated by \( \lambda \), a longitude, and \( \mu \), the boundary of a meridian disc. Let \( C_1 \) and \( C_2 \) be disjoint loops in \( \partial T \), each representing \( 2\lambda + \mu \). There is a Möbius band \( M \) in \( T \) with boundary \( C_2 \). The double of \( T \) is an \( S^2 \times S^1 \) in which \( M \) and the other copy of \( M \) form a 1-sided Klein bottle. The double has a Seifert fibering which is longitudinal on the Klein bottle, nonsingular on its complement, and in which \( C_1 \) is a fiber. If the attaching map in the doubling is changed by Dehn twists about \( C_1 \), the resulting manifolds are of the form \( M(1, n) \), since they still have fiberings which are longitudinal on the Klein bottle and nonsingular on its complement. Since \( \mu \) intersects \( C_1 \) twice, the image of \( \mu \) under \( k \) Dehn twists about \( C_1 \) is \( \mu + 2k(\mu + 2\lambda) = 4k\lambda + (2k+1)\mu \), so the resulting manifold is \( L(4k, 2k+1) = L(4k, 2k-1) \). It must equal \( M(1, k) \) since \( M(1, k) \) is the only manifold of the form \( M(1, n) \) with fundamental group \( C_{4k} \).

As we have seen, with the meridinal fibering the manifolds \( M(m, n) \) have fibers of orders \( 2 \), \( 2 \), and \( m \), so in the terminology of [26], \( M(2, n) \) is a quaternionic manifold, while for \( m > 2 \), \( M(m, n) \) is a (nonquaternionic) prism manifold.

The mapping class groups for the \( M(m, n) \) were calculated in \([3, 6, 31]\), and are given in table 1, in which \( S_3 \) stands for the symmetric group on three letters. As mentioned in the introduction, the full isometry groups \( \text{Isom}(M) \) for all elliptic 3-manifolds were calculated in [26], where it is verified that \( \text{Isom}(M) \to \text{Diff}(M) \) is a bijection on path components. This is the "\( \pi_0 \)"-part of the Generalized Smale Conjecture. What remains is to show that the inclusion induces an isomorphism on all higher homotopy groups, so in the remainder of this paper, we focus only on \( \text{isom}(M) \) and \( \text{diff}(M) \), the connected components of the identity map in \( \text{Isom}(M) \) and \( \text{Diff}(M) \).

Here is an outline of the sections of this paper, where at all times we assume that either \( m > 1 \) or \( n > 1 \). In section 2, we compute the connected components of the identity in the isometry groups of the \( M(m, n) \), and in section 3 we construct certain Seifert fiberings and examine how the groups of isometries acts on them. Also, we introduce "special" Klein bottles, and develop some of their properties. These are needed for section 4, in which we reduce the Generalized Smale Conjecture for our cases to proving that the inclusion from the space of fiber-preserving imbeddings from \( K_0 \) into \( M(m, n) \) to the space of all imbeddings is a weak homotopy equivalence on the connected components of the inclusion. The proof of the latter assertion occupies the final three sections. In fact, we prove there that a parameterized family \( F: D^k \to \text{imb}(K_0, M(m, n)) \) with \( F(t) \) a fiber-preserving imbedding for all \( t \in \partial D^k \) is homotopic, keeping \( F(t) \) fiber-preserving for all \( t \in \partial D^k \), to a family \( G \) with \( G(t) \) fiber-preserving for all \( t \in D^k \).
not appear to be possible to avoid examining imbeddings for which $F(t)(K_0)$ meets $K_0$ non-transversely, but it is enough to consider “generic position” non-transverse configurations, as described in [22]. For these we show in section 5 that there exists a concentric fibered torus $T_u$ (i.e. the boundary of a tubular neighborhood $P_u$ of $K_0$) which meets $F(t)(K_0)$ transversely in circles that are of restricted kinds (see proposition 5.2). In section 6, we show that any parameterized family $F$ with $F(t)$ fiber-preserving for $t \in \partial D^k$, can be deformed relative to $\partial D^k$ so that each $F(t)(K_0)$ meets some level $T_u$ in the restricted kinds of circles. In section 7, we complete the argument, using the methods of Hatcher to simplify the intersections of $F(t)(K_0)$ with the concentric tori $T_u$, then deforming the $F(t)$ to be fiber-preserving inductively over the skeleta of a triangulation of $D^k$. The restrictions on the kinds of circles that can occur in $F(t)(K_0) \cap T_u$ ensure that after Hatcher’s process is carried out, the intersection circles that remain in $F(t)(K_0) \cap T_u$ are isotopic in $T_u$ to fibers.

Our program fails in a fundamental way in the case of $L(4, 1)$ because no Seifert fibering of $L(4, 1)$ that contains a fibered Klein bottle is preserved by all isometries. As we proceed through the argument, we will point out more precisely the steps that break down for $L(4, 1)$.

### 2. Isometries

The finite subgroups of SO(4) that act freely on $S^3$ were determined by Hopf and Seifert-Threlfall, and reformulated using quaternions by Hattori. References include [43] (pp. 226-227), [27] (pp. 103-113), [35] (pp. 449-457), [34], and [26]. As already mentioned, the latter gives a calculation of the isometry groups of all elliptic 3-manifolds, but since we need to introduce a certain amount of notation anyway, we will recalculate isom($M(m, n)$).

Fix coordinates on $S^3$ as $\{(z_0, z_1) \mid z_i \in \mathbb{C}, z_0\overline{z_0} + z_1\overline{z_1} = 1\}$. Its group structure as the unit quaternions can then be given by writing points in the

<table>
<thead>
<tr>
<th>$m, n$ values</th>
<th>$M(m, n)$</th>
<th>$\mathcal{H}(M(m, n))$</th>
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<tbody>
<tr>
<td>$m = n = 1$</td>
<td>$L(4, 1)$</td>
<td>$C_2$</td>
</tr>
<tr>
<td>$m = 1, n &gt; 1$</td>
<td>$L(4n, 2n - 1)$</td>
<td>$C_2 \times C_2$</td>
</tr>
<tr>
<td>$m = 2, n = 1$</td>
<td>quaternionic manifold</td>
<td>$S_3$</td>
</tr>
<tr>
<td>$m = 2, n &gt; 1$</td>
<td>quaternionic manifold</td>
<td>$S_3 \times C_2$</td>
</tr>
<tr>
<td>$m &gt; 2, n = 1$</td>
<td>prism manifold</td>
<td>$C_2$</td>
</tr>
<tr>
<td>$m &gt; 2, n &gt; 1$</td>
<td>prism manifold</td>
<td>$C_2 \times C_2$</td>
</tr>
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</table>

Table 1. Mapping class groups of the $M(m, n)$
form $z_0 + z_1j$, where $j^2 = -1$ and $jz_i = z_1j$. The unique element of order 2 in $S^3$ is $-1$, and it generates the center of $S^3$.

By $S^1$ we will denote the subgroup of points in $S^3$ with $z_1 = 0$, that is, all quaternions of the form $z_0$, where $z_0$ lies in the unit circle in $\mathbb{C}$. Let $\xi_k = \exp(2\pi i/k)$, which generates a cyclic subgroup $C_k \subset S^1$. The elements $S^1 \cup S^1 j$ form a subgroup $O(2)^* \subset S^3$, which is exactly the normalizer of $C_k$ if $k > 2$. Also contained in $O(2)^*$ is the binary dihedral group $D_{4m}^*$ generated by $x = j$ and $y = \xi_{2m}$; its normalizer is $D_{8m}^*$. By $J$ we denote the subgroup of $S^3$ consisting of the elements with both $z_0$ and $z_1$ real. It is the centralizer of $j$.

The usual inner product on $S^3$ is given by $z \cdot w = \Re(zw^{-1})$, where $\Re(z_0 + z_1j) = \Re(z_0)$. Consequently, left multiplication and right multiplication by elements of $S^3$ are orthogonal transformations of $S^3$, and there is a homomorphism $F: S^3 \times S^3 \to SO(4)$ defined by $F(q_1, q_2)(q) = q_1 q q_2^{-1}$. It is surjective and has kernel $\{(1, 1), (-1, -1)\}$.

Suppose $G$ is a finite subgroup of $SO(4)$ acting freely on $S^3$. Since $SO(4)$ is the full group of orientation-preserving isometries of $S^3$, the orientation-preserving isometries $\text{Isom}_+(S^3/G)$ are the quotient $\text{Norm}(G)/G$, where $\text{Norm}(G)$ is the normalizer of $G$ in $SO(4)$. Assuming that the group $G$ is clear from the context, we denote the isometry that an element $F(q_1, q_2)$ of $\text{Norm}(G)$ induces on $S^3/G$ by $f(q_1, q_2)$.

Let $G^* = F^{-1}(G)$, and let $G_L$ and $G_R$ be the projections of $G^*$ into the left and right factors of $S^3 \times S^3$. Notice that $\text{Norm}(G)/G \cong \text{Norm}(G^*)/G^*$. The connected component of the identity in $\text{Norm}(G^*)$ is denoted by $\text{norm}(G^*)$. It is the product $\text{norm}(G_L) \times \text{norm}(G_R)$ of the corresponding connected normalizers of $G_L$ and $G_R$ in the $S^3$ factors. The connected component of the identity in the isometry group of $S^4/G$ is then $\text{isom}(M) = \text{norm}(G^*)/(\text{norm}(G^*) \cap \text{norm}(G^*))$.

We now use the previous description to compute $\text{isom}(M(m, n))$.

Case I. $m = 1$.

In order that the fibering we later want to use on $M(1, n)$ be the one induced by the Hopf fibering of $S^3$, we are going to imbed $\pi_1(M(1, n))$ into $SO(4)$ in an unusual way. First, note that the element $F(\xi^{n-1}_{4n}, i)$ acts on $S^3$ by

$$F(\xi^{n-1}_{4n}, i)(z_0 + z_1j) = \xi^{n-1}_{4n} z_0(-i) + \xi^{n-1}_{4n} z_1 j(-i) = \xi^{-1}_{4n} z_0 + \xi^{2n-1}_{4n} z_1 j.$$ 

Consequently the quotient of $S^3$ by the subgroup generated by $F(\xi^{n-1}_{4n}, i)$ is $L(4n, 2n + 1) = L(4n, 2n - 1) = M(1, n)$. Conjugation by $F(1, \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j)$ moves $F(\xi^{n-1}_{4n}, i)$ to $F(\xi^{n-1}_{4n}, j)$. This will be our standard generator for $G = \pi_1(M(1, n))$.

Observe that $G_R$ is the cyclic subgroup of order 4 generated by $j$, so $\text{norm}(G_R) = J$, and $G_L$ is $\langle \xi^{n-1}_{4n}, -1 \rangle$. If $n = 1$, then $\xi^{n-1}_{4n} = 1$ and $G_L = C_2$. If $n > 1$ then $\xi^{n-1}_{4n}$ has order $4n/(4n, n - 1)$, so $G_L$ is $C_{2n}$ if $n$ is odd and $C_{4n}$ if $n$ is even.
Table 2. Isometry groups of the $M(m,n)$

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<tr>
<th>$m,n$ values</th>
<th>$M$</th>
<th>$\text{isom}(M)$</th>
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<tbody>
<tr>
<td>$m=n=1$</td>
<td>$L(4,1)$</td>
<td>$SO(3) \times S^1 = { f(q, x) \mid (q, x) \in S^3 \times J }$</td>
</tr>
<tr>
<td>$m=1, n&gt;1$</td>
<td>$L(4n, 2n-1)$</td>
<td>$S^1 \times S^1 = { f(x, y) \mid (x, y) \in S^1 \times J }$</td>
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<td>$m&gt;1, n=1$</td>
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<td>$S^3 = { f(x, 1) \mid x \in S^1 }$</td>
</tr>
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(1) If $n = 1$, then $\text{norm}(G_L) = S^3$, and $\text{isom}(M(1,1)) \cong SO(3) \times S^1$, consisting of all isometries of the form $f(q, x)$ with $(q, x) \in S^3 \times J$.

(2) If $n > 1$, then $\text{norm}(G_L) = S^1$, so $\text{isom}(M(1, n)) \cong S^1 \times S^1$, consisting of all isometries of the form $f(x_1, x_2)$ with $(x_1, x_2) \in S^1 \times J$.

Case II. $n = 1$ and $m > 1$.

We imbed $G = D^*_{4m}$ in $SO(4)$ as the subgroup $F(D^*_{4m} \times \{1\})$. We have $G_L = D^*_{4m}$ and $G_R = C_2$, so $\text{norm}(G_L) \times \text{norm}(G_R) = \{1\} \times S^3$. Therefore $\text{isom}(M(m,1)) \cong SO(3)$, and consists of all isometries of the form $f(1, q)$.

Case III. $m > 1$ and $n > 1$.

If $n$ is odd, then $G = C_n \times D^*_{4m}$, and we imbed $G$ in $SO(4)$ as $F(C_{2n} \times D^*_{4m})$, so $G_L = C_{2n}$ and $G_R = D^*_{4m}$. If $n$ is even, then $G$ is the image in $SO(4)$ of the unique diagonal subgroup of index 2 in $C_{4n} \times D^*_{4m}$, so $G_L = C_{4n}$ and $G_R = D^*_{4m}$. In either case, we have $\text{norm}(G_L) \times \text{norm}(G_R) = S^1 \times \{1\}$. Therefore $\text{isom}(M(m,n)) \cong S^1$, and consists of all isometries of the form $f(x, 1)$ with $x \in S^1$.

Table 2 summarizes our calculations of $\text{Isom}(M(m,n))$.

3. Seifert fiberings and special Klein bottles

From now on, we use $M$ to denote one of the manifolds $M(m,n)$ with $m > 1$ or $n > 1$. In this section, we construct certain Seifert fiberings of these $M$, and examine the effect of $\text{isom}(M)$ on them. Also, for these fiberings we define certain special Klein bottles in $M$, which play an important role in the reduction carried out in section 4.

We will regard the 2-sphere $S^2$ as $\mathbb{C} \cup \{\infty\}$. We speak of antipodal points and orthogonal transformations on $S^2$ by transferring them from the unit 2-sphere using the stereographic projection that identifies the point $(x_1, x_2, x_3)$ with $(x_1 + x_2i)/(1 - x_3)$. For example, the antipodal map $\alpha$ is defined by $\alpha(x) = -1/\overline{x}$.

As is well-known, the Hopf fibering on $S^3$ is an $S^1$-bundle structure with projection map $H: S^3 \to S^2$ defined by $H(z_0, z_1) = z_0/z_1$. The left action of $S^1$ on $S^3$ takes each Hopf fiber to itself, so preserves the Hopf fibering. The element $F(j, 1)$ also preserves it. For $j(z_0 + z_1j) = -\overline{z_1} + \overline{z_0}j$, we have $F(j, 1)$.
so $H(F(j,1)(z_0 + z_1j)) = -1/z_0/z_1$. Since right multiplication by elements of $S^3$ commutes with the left action of $S^1$, it preserves the Hopf fibering, and there is an induced action of $S^3$ on $S^2$. In fact, it acts orthogonally. For if we write $x = x_0 + x_1j$ and $z = z_0 + z_1j$, we have $zx^{-1} = z_0x_0^* + z_1x_1^* + (z_1x_0 - z_0x_1)j$, so the induced action on $S^2$ carries $z_0/z_1$ to $(z_0x_0^* + z_1x_1^*(z_1x_0 - z_0x_1)) = \left( \frac{x_0}{-x_1} \frac{x_1}{x_0} \right) (z_0/z_1)$. The trace of this linear fractional transformation is real and lies between $-2$ and $2$ (unless $x = \pm 1$, which acts as the identity on $S^2$), so it is elliptic. Its fixed points are $(x_0 - \overline{x_0}) \pm \sqrt{(x_0 - \overline{x_0})^2 - 4x_1\overline{x_1}}/(2x_1)$, which are antipodal, so it is an orthogonal transformation. Combining these observations, we see that the action induced on $S^2$ via $H$ determines a surjective homomorphism $h: O(2^*) \times S^3 \to O(3)$, given by $h(x_0,1) = 1$ for $x_0 \in S^1$, $h(j,1) = \alpha$, and $h(1,x_0 + x_1j) = \left( \frac{x_0}{-x_1} \frac{x_1}{x_0} \right)$. The kernel of $h$ is $S^1 \times \{ \pm 1 \}$.

Since each of our groups $G = \pi_1(M)$ lies in $F(O(2^*) \times S^3)$, it preserves the Hopf fibering, which descends to a Seifert fibering on $S^3/G$ called the Hopf fibering. Its quotient orbifold is $S^2/h(G)$, and the orbit map is induced by the composition of $H$ followed by the quotient map from $S^2$ to $S^2/h(G)$. The quotient orbifolds for our fiberings are easily calculated using the explicit imbeddings of $G$ into $SO(4)$ given in section 2, together with the facts that $h(j,1) = \alpha$, $h(1,\xi_{2m}) = r_m$, the (clockwise) rotation through an angle $2\pi/m$ with fixed points 0 and $\infty$, defined by $r_m(z) = \xi^{-1}_m z$, and $h(1,j) = t$, the rotation through an angle $\pi$ with fixed points $\pm i$, defined by $t(z) = -1/z$.

Table 3 lists the various cases, where $(F; n_1, \ldots, n_k)$ denotes the 2-orbifold with underlying topological space the surface $F$ and $k$ cone points of orders $n_1, \ldots, n_k$.

Since $m > 1$ or $n > 1$, we have $\text{norm}(\pi_1(M)) \subset F(O(2^*) \times S^3)$, so $\text{isom}(M)$ preserves the Hopf fibering. Since the quotient orbifolds are the quotients of orthogonal actions on $S^2$, they have metrics of constant curvature 1, except at the cone points, where the cone angle at an order $k$ cone point is $2\pi/k$. Table 3 shows the quotient orbifolds with shapes that suggest the symmetries for this constant curvature metric. The isometry group of each orbifold $O$ is the normalizer of its orbifold fundamental group $h(G)$ in the isometry group $O(3)$ of $S^2$. The homomorphism $h$ induces a homomorphism $\text{isom}(M) \to \text{isom}(O)$, and from the explicit description of $\text{isom}(M)$ from table 2 we can use $h$ to compute the image. In each case, all isometries in the connected component of the identity, $\text{isom}(O)$, are induced by elements of $\text{isom}(M)$. (The groups $\text{isom}(O)$ are computed as $\text{norm}(G)/(G \cap \text{norm}(G))$ where $\text{norm}(G)$ is the connected component of the identity in the normalizer of $G$ in $\text{isom}(S^2) = SO(3)$. In particular, $\text{isom}(\mathbb{R} P^2) = SO(3)$, which can be seen directly by noting that each isometry of $\mathbb{R} P^2$ lifts to an unique orientation-preserving isometry of $S^2$.)
Our next task is to understand the fibered Klein bottles in $M$. We call a torus $T \subset S^3$ *special* if its image in $S^2$ under $H$ is a great circle. They are Clifford tori, that is, they have induced curvature zero in the usual metric on $S^3$. Note that special tori are fibered. Klein bottles in $M$ that are the images of special tori in $S^3$ are called special Klein bottles. It is easy to identify special Klein bottles. A suborbifold in $O$ is called *special* when it is either

(i) a one-sided geodesic circle, or
(ii) a geodesic arc connecting two order-2 cone points.

A Klein bottle in $M$ is special if and only if its image in $O$ is a special suborbifold. For consider a special torus $T$ in $S^3$. If its image in $O$ is special, then its image in $M$ is a one-sided submanifold, so must be a Klein bottle. Conversely, the projection of $T$ to $O$ must always be geodesic, and if its image in $M$ is a submanifold, then the projection to $O$ cannot have any self intersections or meet a cone point of order more than 2. And if the projection is a circle, it is 1-sided if and only if the image of $T$ in $M$ is one-sided.

Note that the fibering on a special Klein bottle is meridinal in case (i), and longitudinal in case (ii). From table 3, we see that:

(1) If $n = 1$ then special Klein bottles always have the meridinal fibering.
(2) If $n > 1$ then special Klein bottles always have the longitudinal fibering.

For each of the quotient orbifolds discussed above, $\text{isom}(O)$ acts transitively on the special suborbifolds. Since all elements of $\text{isom}(O)$ are induced

<table>
<thead>
<tr>
<th>$m, n$ values</th>
<th>$h(\pi_1(M))$</th>
<th>$O$</th>
<th>$\text{isom}(O)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m &gt; 1$, $n = 1$</td>
<td>$C_2 = \langle \alpha \rangle$</td>
<td>$(\mathbb{R}P^2,)$</td>
<td>$\text{SO}(3)$</td>
</tr>
<tr>
<td>$m = 1$, $n &gt; 1$</td>
<td>$C_2 = \langle t \rangle$</td>
<td>$(S^2; 2, 2)$</td>
<td>$\text{SO}(2)$</td>
</tr>
<tr>
<td>$m &gt; 1$, $n &gt; 1$</td>
<td>$D_{2m} = \langle r_m, t \rangle$</td>
<td>$(S^2; 2, 2, m)$</td>
<td>${1}$</td>
</tr>
</tbody>
</table>

Table 3. Quotient orbifolds for the Hopf fiberings
by elements of isom(M), it follows that isom(M) acts transitively on the space of special Klein bottles in M.

Let T₀ be the fibered torus H⁻¹(U), where U is the unit circle in S². Explicitly, T₀ consists of all z₀ + z₁j for which |z₀| = |z₁| = 1/√2. Observe that the isometries F(O(2) × O(2)) of S³ leave T₀ invariant. The action of F(O(2) × O(2)) on T₀ can be calculated using the normalized coordinates [x₀, y₀] ∈ S¹ × S¹/⟨(-1, -1)⟩, where [x₀, y₀] corresponds to the point x₀(1/√2 y₀ i + 1/√2 y₀ j). For (z₀, w₀) ∈ S¹ × S¹, we have F(z₀, w₀)[x₀, y₀] = [z₀x₀, w₀y₀]. Also:

(a) F(j, 1)[x₀, y₀] = [−x₀, i y₀]. Viewed in the fundamental domain Ω(x₀) ≥ 0 for T₀, this rotates the y₀-coordinate through π/2, and reflects in the x₀-coordinate fixing the point i.
(b) F(1, j)[x₀, y₀] = [i x₀, −y₀]. Viewed in the fundamental domain Ω(y₀) ≥ 0 for T₀, this rotates the x₀-coordinate through π/2, and reflects in the y₀-coordinate fixing the point i.

In fact, the restriction of F(O(2) × O(2)) to T₀ is exactly the group of all fiber-preserving isometries Isom₁(T₀). The Hopf fibers are the orbits of the action of F(S¹ × {1}) on T₀, so are the circles with constant y₀-coordinate. Using (a) and (b), we find that that F(j, i)[x₀, y₀] = [x₀, i y₀] and F(i, j)[x₀, y₀] = [x₀, −y₀]. The elements F(z₀, w₀) act transitively on T₀, and only the two reflections F(j, i) and F(j, i) and their composition fix [1, 1] and preserve the fibers, so together they generate all the fiber-preserving isometries.

Since F(O(2) × O(2)) contains (each of our groups) G, the image of T₀ in M is a fibered submanifold K₀. When n = 1, the image of T₀ in O is a geodesic circle which is the center circle of a Möbius band, and when n > 1 its image is a geodesic arc connecting two cone points of order 2, so K₀ is a special Klein bottle. Since G acts by isometries, the subspace metric on T₀ induces a metric on K₀ such that the inclusion of K₀ into M is isometric. According as n = 1 or n > 1, K₀ has either a meridinal or longitudinal fibering for which this inclusion is fiber-preserving. Denote by isom₁(K₀, M) the connected component of the inclusion in the space of all fiber-preserving isometric imbeddings of K₀ into M. Since the isometries of M are fiber-preserving, their compositions with the inclusion determine a map isom(M) → isom₁(K₀, M). The space isom₁(K₀, M) contains isom₁(K₀), the connected component of the identity in the group of fiber-preserving isometries of K₀.

Lemma 3.1. If m > 1 or n > 1, then isom(M) → isom₁(K₀, M) is a homeomorphism. Moreover,

(a) If n = 1, then the elements f(1, w₀) for w₀ ∈ S¹ preserve K₀, and restriction of this subgroup of isom(M) gives a homeomorphism S¹ → isom₁(K₀).
(b) If \( n > 1 \), then the elements \( f(x_0, 1) \) for \( x_0 \in S^1 \) preserve \( K_0 \), and restriction of this subgroup of \( \text{isom}(M) \) gives a homeomorphism \( S^1 \to \text{isom}_f(K_0) \).

Proof. For injectivity, suppose that an element of \( \text{isom}(M) \) fixes each point of \( K_0 \). Since it is isotopic to the identity, it cannot interchange the sides of \( K_0 \). Since it is an isometry, this implies it is the identity on all of \( M \).

For surjectivity, we have seen that \( \text{isom}(M) \) acts transitively on the special Klein bottles. So it remains to check that any element of \( \text{isom}_f(K_0, M) \) that carries \( K_0 \) to \( K_0 \) is the restriction of an element of \( \text{isom}(M) \).

Consider first the case when \( m > 1 \) and \( n = 1 \), so \( G = F(D^*_{4m} \times \{1\}) \) and \( K_0 \) has the meridinal fibering. The fiber-preserving isometry group \( \text{Isom}_f(K_0) \) is \( \text{Norm}(G)/G \) where \( \text{Norm}(G) \) is the normalizer of \( G \) in \( \text{Isom}_f(T_0) \). The elements in \( F(C_{2m} \times \{1\}) \) rotate in the \( x_0 \)-coordinate, while the element \( F(j, 1) \) is as described in (a). So each element of \( G - C_{2k} \) leaves invariant a pair of circles each having constant \( x_0 \)-coordinate. The union of these invariant circles for all the elements of \( G - C_{2k} \) must be invariant under the action of \( \text{Norm}(G) \) on \( T_0 \), so the identity component of \( \text{Norm}(G) \) consists only of \( F(\{1\} \times S^1) \). Consequently the elements \( f(1, w_0) \) of \( \text{isom}(M) \) induce all elements of \( \text{isom}_f(K_0) \), proving the surjectivity of \( \text{isom}(M) \to \text{isom}_f(K_0, M) \) and verifying assertion (a).

For \( m = 1 \) and \( n > 1 \), \( G \) is cyclic generated by \( F(\xi_{4m}^{n-1}, j) \) and \( K_0 \) has the longitudinal fibering. Since \( F(1, j) \) is as described in (b), there is a pair of circles in \( T_0 \), each having constant \( y_0 \)-coordinate and each invariant under all elements of \( G \) (these circles become the fibers that are center circles of Möbius bands in \( K_0 \)). Since these circles must be invariant under the normalizer of \( G \) in \( \text{Isom}(T_0) \), the identity component of \( \text{Norm}(G) \) consists only of \( F(S^1 \times \{1\}) \). Therefore the isometries \( f(x_0, 1) \) with \( x_0 \in S^1 \) of \( \text{isom}(M) \) induce all elements of \( \text{isom}_f(K_0) \), proving the surjectivity of \( \text{isom}(M) \to \text{isom}_f(K_0, M) \) and verifying assertion (a).

Finally, if both \( m > 1 \) and \( n > 1 \), then \( G \) contains \( F(1, j) \) and \( K_0 \) has the longitudinal fibering. Again, the identity component of \( \text{Norm}(G) \) is \( F(S^1 \times \{1\}) \), and the isometries \( f(x_0, 1) \) with \( x_0 \in S^1 \) induce all of \( \text{isom}_f(K_0) \). \( \square \)

In the space of all smooth fiber-preserving imbeddings of \( K_0 \) in \( M \) (for the appropriate fibering on \( K_0 \)), let \( \text{imb}_f(K_0, M) \) denote the connected component of the inclusion.

**Lemma 3.2.** If \( m > 1 \) or \( n > 1 \), then \( \text{isom}_f(K_0, M) \to \text{imb}_f(K_0, M) \) is a weak homotopy equivalence.

Proof. Let \( K_0 \) be the image of \( K_0 \) in the quotient orbifold \( O \) of the fibering on \( M \). As we have seen, when \( K_0 \) has the meridinal fibering, \( K_0 \) is a one-sided geodesic circle in \( O \), and when \( K_0 \) has the longitudinal fibering, \( K_0 \) is a geodesic arc connecting two order 2 cone points of \( O \). Let \( \text{imb}(K_0, O) \) denote the connected component of the inclusion in the space of orbifold imbeddings, and \( \text{isom}(K_0, O) \) its subspace of isometric imbeddings, and let
a subscript $v$ as in $\text{Diff}_v(K_0)$ indicate the vertical maps—those that take each fiber to itself. Consider the following diagram, which we call the main diagram:

$$
\begin{array}{cccccc}
\text{Isom}_v(K_0) \cap \text{isom}_f(K_0) & \longrightarrow & \text{isom}_f(K_0, M) & \longrightarrow & \text{isom}(K_0, \mathcal{O}) \\
\downarrow & & \downarrow & & \downarrow \\
\text{Diff}_v(K_0) \cap \text{diff}_f(K_0) & \longrightarrow & \text{imb}_f(K_0, M) & \longrightarrow & \text{imb}(K_0, \mathcal{O})
\end{array}
$$

in which the vertical maps are inclusions. The left-hand horizontal arrows are inclusions, and the right-hand horizontal arrows take each imbedding to the imbedding induced on the quotient objects. By theorem 8.9 of [24], the bottom row is a fibration. We will now examine the top row.

Suppose first that $n = 1$, so that $\mathcal{O} = (\mathbb{R} \mathbb{P}^2,)$ and $K_0$ is the image of the unit circle $U$ of $S^2$. For this case, $\text{isom}(K_0, \mathcal{O})$ can be identified with the unit tangent space of $\mathbb{R} \mathbb{P}^2$. For if we fix a unit tangent vector of $K_0$, the image of this vector under an isometric imbedding is a unit tangent vector to $\mathbb{R} \mathbb{P}^2$, and each unit tangent vector of $\mathbb{R} \mathbb{P}^2$ corresponds to a unique isometric imbedding of $K_0$. To understand this unit tangent space, note first that the unit tangent space of $S^2$ is $\mathbb{R} \mathbb{P}^3$, since each unit tangent vector to $S^2$ corresponds to a unique element of $\text{SO}(3) = \mathbb{R} \mathbb{P}^3$. The unit tangent space of $S^2$ double covers the unit tangent space of $\mathbb{R} \mathbb{P}^2$, so the latter must be $L(4, 1)$.

Since the isometries of $M$ are all fiber-preserving, there is a commutative diagram

$$
\begin{array}{ccc}
\text{isom}(M) & \longrightarrow & \text{isom}(\mathcal{O}) \\
\rho \downarrow & & \rho \downarrow \\
\text{isom}_f(K_0, M) & \longrightarrow & \text{isom}(K_0, \mathcal{O})
\end{array}
$$

By lemma 3.1, the restriction $\rho$ is a homeomorphism, and from table 3, $h$ is a homeomorphism. The restriction $\rho$ is a 2-fold covering map, since there are two isometries that restrict to the inclusion on $K_0$: the identity and the reflection across $K_0$. This identifies the second map of the top row of the main diagram as the 2-fold covering map from $\mathbb{R} \mathbb{P}^3$ to $L(4, 1)$, with fiber the vertical elements of $\text{isom}_f(K_0)$. We will identify $\text{Isom}_v(K_0) \cap \text{isom}_f(K_0)$ as the fiber of this covering map, by checking that it is $C_2$, generated by the isometry $f(1, i)$. By part (a) of lemma 3.1, the elements of $\text{isom}_f(K_0)$ are induced by the isometries $f(1, w_0)$ for $w_0 \in S^1$. Such an isometry is vertical precisely when $h(1, w_0)$ acts as the identity or the antipodal map on $U$, since each fiber of $K_0$ is the image of the circles in $S^3$ which are the preimages of antipodal points of $U$ (since these are exactly the fibers of $T_0$ that are identified by elements of $G = F(D_4 \times \{1\})$). For $x_0 \in U$, we have $h(1, w_0)(x_0) = \left( \begin{array}{cc} w_0 & 0 \\ 0 & w_0 \end{array} \right) (x_0) = \overline{w_0 x_0}$. So $h(1, w_0)$ is the identity or antipodal map of $U$ exactly when $w_0 = \pm 1$ or $\pm i$. The
cases \( w_0 = \pm 1 \) give \( f(1, 1) \) and \( f(1, -1) \), which are the identity on \( M \) since 
\( F(-1, 1) = F(1, -1) \in G \). Since \( f(1, -1) \) is already in \( G \), \( f(1, i) \) and \( f(1, -i) \) 
are the same isometry on \( K_0 \) and give the unique nonidentity element of 
\( \text{Isom}_v(K_0) \cap \text{isom}_f(K_0) \).

Suppose now that \( m = 1 \). This time, both \( \tilde{\rho} \) and \( \rho \) are homeomorphisms, 
since \( K_0 \) is just a geodesic arc connecting the two order-2 cone points of 
\( \mathcal{O} = (S^2; 2, 2) \). From table 3, \( h: \text{isom}(M) \to \text{isom}(\mathcal{O}) \) is just the projection from 
\( S^1 \times S^1 \) to its second coordinate. The first coordinate is left multiplication 
of \( S^3 \) by elements of \( S^1 \), which by part (b) of lemma 3.1 give exactly the 
elements of \( \text{isom}_f(K_0) \). Since \( h(x_0, 1) \) is the identity on \( S^2 \) for all these \( x_0 \), 
\( \text{isom}_f(K_0) = \text{Isom}_v(K_0) \cap \text{isom}_f(K_0) \). So the top row of the main diagram 
is simply the product fibration \( S^1 \to S^1 \times S^1 \to S^1 \), where the second map 
is projection to the second coordinate. Finally, if both \( m > 1 \) and \( n > 1 \), 
\( \text{isom}(K_0, \mathcal{O}) \) consists of two contractible components, one in which the diffeomorphisms 
preserve the orientation of each fiber and the other in which they reverse it 
(\( \text{Diff}_v(K_0) \cap \text{diff}_f(K_0) \) consists of four contractible components, these two and two others 
represented by the same maps composed with a single Dehn twist about

\[
\begin{array}{c}
\text{Isom}(\mathcal{O} \text{ rel } K_0) \\
\downarrow
\end{array} \begin{array}{c}
\text{isom}(\mathcal{O}) \\
\downarrow
\end{array} \begin{array}{c}
\text{isom}(K_0, \mathcal{O}) \\
\downarrow
\end{array}
\begin{array}{c}
\text{Diff}(\mathcal{O} \text{ rel } K_0) \\
\downarrow
\end{array} \begin{array}{c}
\text{diff}(\mathcal{O}) \\
\downarrow
\end{array} \begin{array}{c}
\text{imb}(K_0, \mathcal{O})
\end{array}
\]

The bottom row is a fibration by the Palais-Cerf Restriction Theorem (corollary 3.7 of [24]), and we have already seen how to identify the top row 
with the covering fibration \( C_2 \to \mathbb{R} \mathbb{P}^3 \to L(4, 1) \). Each component of 
\( \text{Diff}(\mathcal{O} \text{ rel } K_0) \) can be identified with \( \text{Diff}(D^2 \text{ rel } \partial D^2) \), which is contractible 
by [36], so the left vertical arrow is a homotopy equivalence. The middle 
arrow is a homotopy equivalence by the main result of [15]. Consequently 
the right vertical arrow is a weak homotopy equivalence, which is also the 
right vertical arrow of the main diagram.

We have already seen that part (a) of lemma 3.1 identifies \( S^1 \), the group 
of isometries of the form \( f(x_0) \), with \( \text{isom}_f(K_0) \), so that \( f(1, i) \) is the non-
trivial element of \( \text{Isom}_v(K_0) \cap \text{isom}_f(K_0) \). The group \( \text{Diff}_v(K_0) \cap \text{diff}_f(K_0) \) 
consists of two contractible components, one in which the diffeomorphisms 
preserve the orientation of each fiber and the other in which they reverse it 
(\( \text{Diff}_v(K_0) \cap \text{diff}_f(K_0) \) consists of four contractible components, these two and two others 
represented by the same maps composed with a single Dehn twist about
the $a$-circle). The identity map and $f(1, i)$ are points in these two components, so the left vertical arrow of the main diagram is also a homotopy equivalence.

A detailed analysis of $\text{Diff}_v(K_0) \cap \text{imb}_f(K_0)$ can proceed by regarding $K_0$ as a circle bundle over $S^1$, letting $s_0$ be a basepoint in $S^1$ and $C$ be the fiber in $K_0$ which is the preimage of $s_0$, and examining the commutative diagram

\[
\begin{array}{ccc}
\text{Diff}_v(K_0 \text{ rel } C) \cap \text{diff}_f(K_0) & \longrightarrow & \text{Diff}_v(K) \cap \text{diff}_f(K_0) \\
\downarrow & & \downarrow \\
\text{Diff}_f(K_0 \text{ rel } s_0) & \longrightarrow & \text{diff}(S^1) \\
\downarrow & & \downarrow \\
\text{diff}(S^1 \text{ rel } s_0) & \longrightarrow & \text{imb}(s_0, S^1)
\end{array}
\]

whose rows and columns are all fibrations (the first and middle rows by corollary 6.4 of [24], the third row by corollary 3.7 of [24], the first and middle columns by theorem 5.2 of [24], and the third column by theorem 6.6 of [24]. The space in this diagram are homotopy equivalent to the spaces shown here:

\[
\begin{array}{ccc}
\mathbb{Z} & \longrightarrow & C_2 \times \mathbb{R} \\
\downarrow & & \downarrow \\
\mathbb{Z} & \longrightarrow & S^1 \times \mathbb{R} \\
\downarrow & & \downarrow \\
1 & \longrightarrow & S^1
\end{array}
\]

When $n > 1$, the situation is quite a bit simpler. If $m = 1$, $\text{imb}(K_0)$ is just the imbeddings of an arc in $S^1$ relative to two points, which is homotopy equivalent to the subspace $\text{isom}(K_0)$. For the left vertical arrow, $\text{Diff}_v(K_0) \cap \text{diff}_f(K_0)$ has only one component, since a vertical diffeomorphism which reverses the direction of the fibers induces a nontrivial outer automorphism on $\pi_1(K_0)$. The easiest way we know to show that $\text{diff}_v(K_0)$ is homotopy equivalent to a circle is to fix a generic fiber $C$ and a point $c_0$ in $C$, then lift a vertical diffeomorphism to a covering of $K_0$ by $S^1 \times \mathbb{R}$ and equivariantly deform it to the isometry of $S^1 \times \mathbb{R}$ that has the same effect on a lift of $c_0$. This can be carried out canonically using the $\mathbb{R}$-coordinate, so actually gives a deformation retraction to $\text{isom}_v(K_0)$. When $m > 1$, the situation is the same except that $\text{isom}(K_0, \mathcal{O})$ is a point and $\text{imb}(K_0, \mathcal{O})$ is contractible. □

4. Homotopy type of the space of diffeomorphisms

We continue to use the notations of section 3. Our main technical result shows that parameterized families of imbeddings of $K_0$ in $M$ can be deformed to families of fiber-preserving imbeddings:

**Theorem 4.1.** If either $m > 1$ or $n > 1$, then the inclusion $\text{imb}_f(K_0, M) \rightarrow \text{imb}(K_0, M)$ is a weak homotopy equivalence.
Its proof will be given in sections 5, 6, and 7. From theorem 4.1, we can deduce the Generalized Smale Conjecture for our 3-manifolds.

**Theorem 4.2.** If \( m > 1 \) or \( n > 1 \), then the inclusion from \( \text{Isom}(M(m,n)) \) to \( \text{Diff}(M(m,n)) \) is a homotopy equivalence.

**Proofs of theorem 4.2 assuming theorem 4.1.** Since \( \text{Diff}(M) \) has the homotopy type of a CW-complex [29], it is enough to prove that the inclusion is a weak homotopy equivalence. From [26], it is a bijection on path components, so we will restrict attention to the connected components of the identity map.

From corollary 8.7 of [24], restriction of diffeomorphisms to imbeddings defines a fibration

\[
\text{Diff}_v(S^1 \times D^2 \text{ rel } S^1 \times \partial D^2) \to \text{Diff}(S^1 \times D^2 \text{ rel } S^1 \times \partial D^2) \to \text{Diff}(D^2 \text{ rel } \partial D^2),
\]

whose fiber is the group of vertical diffeomorphisms that take each fiber to itself. The base is contractible by [36]. The fiber is contractible, this is seen by lifting diffeomorphisms to the infinite cyclic cover \( \mathbb{R} \times D^2 \) and canonically and equivariantly deforming the lifts to preserve \( \{0\} \times D^2 \), and then to be the identity. We conclude that \( \text{Diff}_f(S^1 \times D^2 \text{ rel } S^1 \times \partial D^2) \) and hence also \( \text{Diff}_f(M \text{ rel } K_0) \) are contractible. Therefore our fibration from above becomes

\[
\text{diff}_f(M \text{ rel } K_0) \longrightarrow \text{diff}_f(M) \longrightarrow \text{imb}_f(K_0, M)
\]

with contractible fiber. Similarly there is a fibration

\[
\text{diff}(M \text{ rel } K_0) \longrightarrow \text{diff}(M) \longrightarrow \text{imb}(K_0, M).
\]

The fact that it is a fibration comes from [28] and the contractibility of the fiber uses [16]. We can now fit these into a diagram

\[
\begin{array}{ccc}
\text{diff}_f(M \text{ rel } K_0) & \longrightarrow & \text{diff}_f(M) & \longrightarrow & \text{imb}_f(K_0, M) \\
\downarrow & & \downarrow & & \downarrow \\
\text{diff}(M \text{ rel } K_0) & \longrightarrow & \text{diff}(M) & \longrightarrow & \text{imb}(K_0, M).
\end{array}
\]

The vertical maps are inclusions. By theorem 4.1, the right hand vertical arrow is a weak homotopy equivalence. Since the fibers are both contractible, it follows that \( \text{diff}_f(M) \to \text{imb}_f(K_0, M) \), \( \text{diff}(M) \to \text{imb}(K_0, M) \), and \( \text{diff}_f(M) \to \text{diff}(M) \) are weak homotopy equivalences.

The right-hand square of the previous diagram is the bottom square of the following diagram, whose vertical arrows are inclusions and whose horizontal
arrows are obtained by restriction of maps to $K_0$:

\[
\begin{array}{ccc}
\text{isom}(M) & \longrightarrow & \text{isom}_f(K_0, M) \\
\downarrow & & \downarrow \\
\text{diff}_f(M) & \longrightarrow & \text{imb}_f(K_0, M) \\
\downarrow & & \downarrow \\
\text{diff}(M) & \longrightarrow & \text{imb}(K_0, M)
\end{array}
\]

From lemma 3.1, $\text{isom}(M) \to \text{isom}_f(K_0, M)$ is a homeomorphism, and from lemma 3.2, $\text{isom}_f(K_0, M) \to \text{imb}_f(K_0, M)$ is a weak homotopy equivalence. We conclude that $\text{isom}(M) \to \text{diff}_f(M)$ is a weak homotopy equivalence, hence so is the composite $\text{isom}(M) \to \text{diff}(M)$. □

5. Generic Position Configurations

Let $S$ and $T$ be smoothly imbedded closed surfaces in a closed 3-manifold $M$. A point $x$ in $S \cap T$ is called a regular point if $S$ is transverse to $T$ at $x$, otherwise it is a singular point. Following section 5 of [22], a singular point $x$ is said to be of finite multiplicity if $S \cap T$ meets a small disc neighborhood $D^2$ of $x$ in $T$ in a finite even number of smooth arcs running from $x$ to $\partial D^2$, which are transverse intersections of $S$ and $T$ except at $x$ (cf. Fig. 3, p. 1653 of [22]). Then, either $S \cap T \cap D^2 = \{x\}$ or $x$ is a saddle tangency of $S$ and $T$. Moreover, if $D^2 \times [-1, 1]$ is a product neighborhood of $x$ which meets $T$ in $D^2 \times \{0\}$, then for some $u_0 > 0$, $S$ meets $D^2 \times \{u\}$ transversely for each $u$ with $0 < |u| \leq u_0$. We say that the surfaces are in generic position if all singular points of intersection are of finite multiplicity.

Now we specialize to the standard Klein bottle $K_0 \subseteq M(m, n)$. To set notation, let $T$ be the torus and fix a 2-fold covering from $T \times [-1, 1]$ to a closed tubular neighborhood of $K_0$, so that $T \times \{0\}$ is a 2-fold covering of $K_0$. For $0 < u < 1$, let $T_u$ denote the image of $T \times \{u\}$, thus $K_0$ together with these $T_u$ form a open tubular neighborhood of $K_0$. We call the $T_u$ levels. Our fixed presentation of $\pi_1(K_0)$ determines elements $a$ and $b^2$ which generate the free abelian group $\pi_1(T_u)$ for each $u > 0$.

Each $T_u$ is the boundary of a tubular neighborhood $P_u$ of $K_0$, and also bounds the solid torus $\overline{M - P_u}$, which we denote by $R_u$. By a meridian in $T_u$ we mean a simple loop in $T_u$ which is essential in $T_u$ but contractible in $R_u$. The meridians represent $(a^m b^{2n})^{\pm 1}$ in $\pi_1(T_u)$. By a longitude in $T_u$ we mean a simple loop in $T_u$ which represents a generator of the infinite cyclic group $\pi_1(R_u)$. The longitudes represent elements of $\pi_1(T_u)$ of the form $(a^n b^q (a^m b^{2n})^k)^{\pm 1}$, where $pn - qm = 1$, since these are precisely the elements whose intersection number with the meridians is $\pm 1$. This leads us to the following observation.
Lemma 5.1. Let $\ell$ be a loop in $T_u$ which represents $a$ or $b^2$ in $\pi_1(T_u)$. Then $\ell$ is not a meridian of $T_u$. If $(m,n) \neq (1,1)$, and $\ell$ is a longitude of $R_u$, then $\ell$ is isotopic in $T_u$ to a fiber of the Seifert fibering of $M(m,n)$.

Proof. Since neither of $m$ nor $n$ is 0, $\ell$ cannot be a meridian. Assume that $(m,n) \neq (1,1)$. If $n=1$, then the fiber is $a^{\pm 1}$ and the longitudes are of the form $(a^{m}b^{2k})^{\pm 1}$. Since in this case $m > 1$, a longitude can never equal $b^{\pm 2}$. Suppose now that $n > 1$. The fiber represents $b^{\pm 2}$ and the longitudes are of the form $(a^{p}b^{2q}(a^{m}b^{2n})^{k})^{\pm 1}$, where $pn - qm = 1$. If a longitude represents $a^{\pm 1}$, then $q + km = 0$. But $q$ and $n$ are relatively prime, and $n > 1$, so this is impossible. 

The lemma fails for $M(1,1)$, for in that case an $a$ circle is a longitude of $R_u$ which is not homotopic to a fiber of the longitudinal fibering, while a $b^2$ circle is a longitude not homotopic to a fiber of the meridinal fibering.

If $K$ is a Klein bottle in $M$ that meets $K_0$ in generic position, then the intersection of $K$ with the nearby levels is restricted by the next proposition, which is the main result of this section.

Proposition 5.2. Suppose that $M = M(m,n)$ with $(m,n) \neq (1,1)$, and let $K$ be a Klein bottle meeting $K_0$ in generic position. Then there exists $u_0 > 0$ so that for each $u < u_0$, $K$ is transverse to $T_u$, and each circle of $K \cap T_u$ is either inessential in $T_u$, or represents $a$ or $b^2$ in $\pi_1(T_u)$.

In order to prove proposition 5.2, we introduce a special kind of isotopy. Suppose that $L_0$ is a 1-sided surface in a closed 3-manifold $N$, and as above let $L_u$ denote the horizontal levels of a tubular neighborhood of $L_0$. A piecewise-linearly imbedded surface $S$ in $N$ is said to be flattened (with respect to $L_0$ and the choice of the $L_u$) if it satisfies the following conditions.

1. There is a 4-valent graph $\Gamma$ (possibly with components which are circles) contained in $L_0$ such that $S \cap L_0$ consists of the closures of some of the connected components of $L_0 - \Gamma$.

2. Each point $p$ in the interior of an edge of $\Gamma$ has a neighborhood $U$ for which the quadruple $(U, U \cap L_0, U \cap S, p)$ is PL homeomorphic to the configuration $(\mathbb{R}^3, \{(x,y,z) | z = 0\}, \{(x,y,z) | \text{either } z = 0 \text{ and } x \geq 0, \text{ or } x = 0 \text{ and } z \geq 0\}, \{0\})$ (see Figure 1(a)).

3. Each vertex $v$ of $\Gamma$ has a neighborhood $U$ for which the quadruple $(U, U \cap L_0, U \cap S, v)$ is PL homeomorphic to the configuration $(\mathbb{R}^3, \{(x,y,z) | z = 0\}, \{(x,y,z) | \text{either } z = 0 \text{ and } xy \leq 0, \text{ or } x = 0 \text{ and } z \geq 0, \text{ or } y = 0 \text{ and } z \leq 0\}, \{0\})$ (see Figure 1(b)).

Lemma 5.3. Let $S_0$ be a smoothly imbedded surface in $N$ which meets $L_0$ in generic position. Then for some $u_0 > 0$, there is a PL isotopy $S_t$ from $S_0$ to a PL imbedded surface $S_1$ such that

1. each $S_t$ is transverse to $L_u$ for $0 < u \leq u_0$, and
2. $S_1$ is flattened.
Proof. By the properties of generic position, there is a $u_0 > 0$ so that $S_0$ is transverse to $L_u$ for all $0 < u \leq u_0$. The isotopy will move points monotonically with respect to $u$ levels. We first describe it near a singular point $x$ of $S_0 \cap L_0$. In a neighborhood $U$ of $x$, $S_0 \cap L_0$ consists of $x$ together with a (possibly empty) collection of arcs $\alpha_1, \alpha_2, \ldots, \alpha_{2n}$ emanating from $x$. There is a neighborhood of $x$ for which the angle of intersection of $S_0$ with $L_0$ is small; the isotopy moves points only within a small neighborhood of the $\alpha_i$ and decreases these angles to 0 everywhere in a neighborhood of $x$ (or pushes a 2-disc neighborhood of $x$ in $S_0$ down to a 2-disc neighborhood of $x$ in $L_0$, if there are no arcs). At the end of the initial isotopy, say for $1 \geq t \geq 1/2$, there is a neighborhood $U$ of $x$ for which $S_{1/2} \cap L_0 \cap U$ is a regular neighborhood in $L_0$ of $\cup_{i=1}^{2n} \alpha_i$. These isotopies may be performed simultaneously near each singular point of intersection. The remainder of the isotopy will move points only in a small neighborhood of the original (open) edges of $S_0 \cap L_0$. At the end of this isotopy, the intersection will be locally a regular neighborhood of the original edges, except that on some of the edges it might be necessary to introduce a point where the configuration is as in Figure 1(b)— this is necessary only when the flattenings at the singular points at the ends of the edge are in opposite senses. Again, these remaining isotopies may be performed simultaneously in disjoint neighborhoods of the original edges.

We call an isotopy as in lemma 5.3 a flattening isotopy. By property (i) of the lemma, the collection of intersection circles in $L_u$ for $0 < u \leq u_0$ is changed only by isotopy in $L_u$. After flattening, each of these circles projects through $S_1$ to an immersed circle lying in $\Gamma$, having a transverse self-intersection at each of its double points (which can occur only at vertices of $\Gamma$.)

Proof of proposition 5.2. Suppose first that the intersection $K \cap K_0$ is transverse. Since $K$ must meet every nearby level $T_u$ transversely, it intersects $P_u$ in Möbius bands and annuli. Consequently the projection of $T_u$ onto $K_0$ maps circles of intersection of $K \cap T_u$ onto circles of $K \cap K_0$ either homeomorphically or by two-fold coverings. Only inessential and $a$ and $b^2$ circles can be preimages of imbedded circles in $K_0$. For suppose a loop representing
Figure 2. Removal of a 2-gon by isotopy

\[ a^k b^{2\ell} \] covers an imbedded circle. Then it must have zero intersection number with its image under the covering transformation \( \tau \) of \( T_u \) over \( K_0 \). Since \( a \) and \( b^2 \) have intersection number 1 in \( T_u \), and \( \tau(a) = a^{-1} \) and \( \tau(b^2) = b^2 \), the image represents \( a^{-k} b^{2\ell} \) and the intersection number is \( 2k\ell \).

Suppose now that \( K \cap K_0 \) contains singular points. By lemma 5.3, we can flatten \( K \) near \( K_0 \), without changing the isotopy classes in \( T_u \) of the loops \( K \cap T_u \). After the flattening, \( K \cap K_0 \) consists of a valence 4 graph \( \Gamma \), which is the image of the collection of disjoint simple closed curves \( K \cap T_u \) under a 2-fold covering projection, together with some of the complementary regions of \( \Gamma \) in \( K_0 \), which we will call the faces. Each edge of \( \Gamma \) lies in the closure of exactly one face. It is convenient to choose an I-fibering of \( P_{u_0} \) so that \( K \cap P_{u_0} \) lies in the union of \( K \cap K_0 \) and the I-fibers that meet \( \Gamma \).

Suppose for contradiction that one of the circles in \( K \cap T_u \) represents \( a^k b^{2\ell} \) with \( k\ell \neq 0 \). Since \( K \) is geometrically incompressible (if not, then \( M \) would contain an imbedded projective plane), there is an isotopy of \( K \) in \( M \) which eliminates the circles of \( K \cap T_u \) that are inessential in \( T_u \), without altering the remaining circles or destroying the flattened position of \( K \cap P_u \). So we may assume that \( K \cap T_u \) consists of disjoint circles each representing \( a^k b^{2\ell} \).

Since \( K \) is isotopic to \( K_0 \), each loop in \( T_u \) has even algebraic intersection number with \( K \cap T_u \), so there is an even number of these circles; denote them by \( A_1, A_2, \ldots, A_{2r} \). Notice that at this point none of the components of \( \Gamma \) can be a circle, since it would then project along I-fibers of \( P_{u_0} \) onto a circle of \( K_0 \cap K \), but we have seen that only inessential, \( a \), and \( b^2 \) circles can cover imbedded circles in \( K_0 \).

The vertices of \( \Gamma \) are the images of the intersections of \( \cup A_i \) with \( \cup \tau(A_i) \) (note that by the properties of \( \Gamma \), \( \cup A_i \) and \( \cup \tau(A_i) \) meet transversely). As above, we compute the intersection number to be

\[
(\cup A_i) \cdot (\cup \tau(A_i)) = (2r a^k b^{2\ell}) \cdot (2r a^{-k} b^{2\ell}) = 4r^2 2k\ell.
\]

Since \( (\cup A_i) \cup (\cup \tau(A_i)) \) is \( \tau \)-invariant, each vertex of \( \Gamma \) is covered by two intersections, so \( \Gamma \) has at least \( 4r^2 |k\ell| \) vertices.

We claim that each edge of \( \Gamma \) runs between two distinct vertices of \( \Gamma \). Supposing to the contrary, we would see a crossing configuration as figure 1(b), for which starting at the origin and traveling along one of the four edges of \( \Gamma \)
that meet there returns to the origin along one of the other three edges without passing through another vertex. Suppose, for example, that the edge is the positive $y$-axis in figure 1(b). Start at a point of the positive $z$-axis and travel through $K$, slightly above the positive $y$-axis. Looking toward $K_0$, one sees a region of $K \cap K_0$ on the left. After traveling around the edge, one returns in the portion of $K$ above (for the negative $y$-axis) or below (for the positive or negative $x$-axis). For any of these three cases, looking toward $K \cap K_0$ one sees the region of $K \cap K_0$ on the right, a contradiction.

More generally, suppose that a face of $K \cap K_0$ contains an odd number of edges. Start as before on the positive $z$-axis at one of the crossings as in figure 1(b), and travel along the lifts of the edges of the face to some $T_u$. At each vertex of $\Gamma$, we cross over from the positive $z$-axis to the negative $z$-axis, or vice versa, and go from seeing the face on the left to seeing it on the right, or vice versa. If the face has an odd number of edges, then we return to the original vertex seeing the face on the wrong side, a contradiction.

Consider a face that is a 2-gon. Since no edge has equal endpoints, the face must have two distinct vertices, as in figure 2(a). The isotopy of $K$ indicated in figure 2 combines this 2-gon with one or two other faces. Repeating, we move $K$ by isotopy (not changing the isotopy classes of the loops of $K \cap T_u$) to eliminate all faces that are 2-gons. So we may assume that each face contains at least 4 vertices.

The Euler characteristic of $K \cap P_u$ is at least $-2r$, since $\chi(K) = 0$ and $K \cap P_u$ has exactly 2r boundary components. Letting $V$, $E$, and $F$ denote the number of vertices, edges, and faces of $K \cap K_0$, we have $E = 2V$ and $F \leq V/2$ (since each edge lies in exactly one face and each face has at least 4 edges). Therefore $-2r \leq \chi(K \cap P_u) = \chi(K \cap K_0) \leq -V/2$ (note that the latter estimate does not require that the faces themselves have Euler characteristic 1). Since $V \geq 4r^2|k\ell|$, it follows that $r|k\ell| \leq 1$, forcing $r = |k\ell| = 1$, $\chi(K \cap K_0) = -2$, $V = 4$, and $F = 2$. That is, $K \cap K_0$ consists of two faces, each a 4-gon, meeting at their four vertices. Since $|k\ell| = 1$, $\Gamma$ is the image of two imbedded circles of $T_u$ each representing $a^{\pm 1}b^{\pm 2}$. Since $\chi(K \cap P_u) = \chi(K \cap K_0) = -2$ and $\chi(K) = 0$, each of these circles must bound a disc in $R_u$. This contradicts the hypothesis that $(m, n) \neq (1, 1)$. \qed

Figure 3 shows $K \cap K_0$ for a Klein bottle $K$ in $M(1, 1)$ that is the flattening of a Klein bottle that meets every $T_u$ close to $K_0$ in longitudes not homotopic to fibers, i.e. in loops representing $ab^2$. This results in a fatal ambiguity in the process used in section 7 below, when one tries to clear such annuli out of $R_u$.

6. Generic position families

Adapting lemma (5.2) and remark (5.3) of [22] to the case at hand, we have:
Proposition 6.1. Let $M = M(m, n)$, and let $F: D^k \to \text{Imb}(K, M)$ be a parameterized family of Klein bottles in $M$. Then by an arbitrarily small deformation, $F$ is homotopic to a family $G: D^k \to \text{Imb}(K, M)$ so that $G(t)(K)$ is in generic position with respect to $K_0$ for all $t \in D^k$.

Note that any imbedded Klein bottle in $M$ must meet $K_0$, since otherwise it would be imbedded in the open solid torus $M - K_0$, so would admit an imbedding into 3-dimensional Euclidean space.

Theorem 6.2. Suppose that $M = M(m, n)$ with $(m, n) \neq (1, 1)$, and let $F: D^k \to \text{Imb}(K, M)$ be a parameterized family of Klein bottles in $M$. Assume that if $t \in \partial D^k$, then $F(t)$ is fiber-preserving and $F(t)(K) \neq K_0$. Then $F$ is homotopic relative to $\partial D^k$ to a family $G$ such that for each $t \in D^k$, there exists $u > 0$ so that $G(t)(K)$ meets $T_u$ transversely and each circle of $G(t)(K) \cap T_u$ is either inessential in $T_u$, or represents $a$ or $b^2$ in $\pi_1(T_u)$.

Proof. Fix a parameter $t \in \partial D^k$, and let $f$ be the composition of $F(t)$ with projection to the $u$-coordinate of $P_1 - K_0$ (so $f$ is defined on a subset of $K$). We noted that $F(t)(K)$ must meet $K_0$, and by hypothesis, $F(t)(K)$ does not equal $K_0$, so the image of $f$ contains an interval. By Sard’s Theorem, almost all values of $u$ are regular values of $f$, so there is some level $T_u$ such that $F(t)(K)$ meets $T_u$ transversely. Since transversality is an open condition and $\partial D^k$ is compact, there are finitely many open sets in $D^k$ whose union contains $\partial D^k$ and such that on each open set, there is a level $T_u$ such that $F(t)(K)$ meets $T_u$ transversely for every $t$ in the open set. At points of $\partial D^k$, the intersection curves are fibers, so must be either $a$- or $b^2$-circles in $\pi_1(T_u)$.

Choose a collar neighborhood $U$ whose closure is contained in the union of these open sets. Again by compactness, there is an $\epsilon$ so that for all $t \in U$, any imbedding within distance $\epsilon$ of $F(t)$ is transverse to one of these levels and its image intersects the level in loops representing either $a$ or $b^2$.

Apply proposition 6.1 to obtain a deformation of $F$, moving each $F(t)$ a distance less than $\epsilon$, to a map $G'$ so that $G'(t)(K)$ meets $K_0$ in generic
position for every $t \in D^k$. We taper this deformation off in the collar neighborhood $U$, obtaining a deformation relative to $\partial D^k$ from $F$ to a map $G$ which agrees with $G'$ on $D^k - U$ and such that for each $t \in U$, $G(t)$ lies within distance $\epsilon$ of $F(t)$. By the property of $\epsilon$, each $G(t)(K)$ for $t \in U$ meets some $T_u$ transversely in loops representing $a$ or $b^2$. For $t \in D^k - U$, each $G(t)(K)$ is in generic position with $K_0$. By proposition 5.2, these $G(t)(K)$ meet all $T_u$, for $u$ sufficiently close to 0, transversely in loops which are either inessential in $T_u$, or represent $a$ or $b^2$ in $\pi_1(T_u)$. \hfill $\square$

7. Parameterization

We now complete the proof of theorem 4.1. By definition, $\text{imb}(K_0, M)$ and $\text{imb}_f(K_0, M)$ are connected, so $\pi_0(\text{imb}(K_0, M), \text{imb}_f(K_0, M)) = 0$. To prove that the higher relative homotopy groups vanish, we begin with a parameterized family, which we may take to be a smooth map $F: D^k \to \text{imb}(K_0, M)$, where $k \geq 1$, which takes all points of $\partial D^k$ to $\text{imb}_f(K_0, M)$.

We will deform it, possibly changing the imbeddings at parameters in $\partial D^k$ but retaining the property that they are fiber-preserving, to a family which is fiber-preserving at every parameter. In fact, all deformations will be relative to $\partial D^k$, except for the first step, given in the next paragraph.

In order to apply theorem 6.2, we must ensure that no $F_t(K_0)$ equals $K_0$ for $t \in \partial D^k$. The image of $\partial D^k$ in $\text{imb}_f(K_0, M)$ is compact, while the space of vertical Klein bottles in $M$ isotopic to $K_0$ is noncompact (topologized using the Hausdorff metric), so we may find a smooth isotopy $J_s$ of $M$, $0 \leq s \leq 1$, with the following properties:

(a) $J_0$ is the identity of $M$.
(b) Each $J_s$ is fiber-preserving.
(c) $J_1(K_0) \neq F(t)(K)$ for any $t \in \partial D^k$.

A deformation of $F$ is now defined by $F_s(t) = J_s^{-1} \circ F(t)$. For $s = 1$, $F_1(t)(K_0) \neq K_0$, since otherwise we would have $J_1(K_0) = J_1F_1(t)(K_0) = F(t)(K_0)$. Since each $J_s$ is fiber-preserving, $F_1$ represents the same element of $\pi_1(\text{imb}(K_0, M), \text{imb}_f(K_0, M))$ that $F$ did, so we may assume that $F(t)(K_0) \neq K_0$ for $t \in \partial D^k$.

For $t \in D^k$, write $K_t$ for $F(t)(K_0)$. Using theorem 6.2, we further deform $F$ so that for each $t$, there is a value $u > 0$ so that

(1) $K_t$ is transverse to $T_u$.
(2) Every circle of $K_t \cap T_u$ is either inessential in $T_u$, or represents either $a$ or $b^2$ in $\pi_1(T_u)$.

The next step is to get rid of inessential intersections. Consider a single $K_t$ and its associated level $T_u$. Notice first that each circle $c$ of $K_t \cap T_u$ that bounds a (necessarily unique) 2-disc $D_T(c)$ in $T_u$ also bounds a unique 2-disc $D_K(c)$ in $K_t$, since $K_t$ is geometrically incompressible. We claim that if $D_K(c)$ is innermost among all such discs on $K_t$, then the interior of $D_K(c)$ is disjoint from $T_u$. If not, then there is a smaller disc $E$ in $D_K(c)$ such that $\partial E$ is essential in $T_u$ and the interior of $E$ is disjoint from $T_u$. Now $E$
cannot be contained in \( P_u \), since \( T_u \) is incompressible in \( P_u \), so \( E \) must be a meridian disc of \( R_u \). But then, \( \partial E \) is a circle of \( K_t \cap T_u \) which is a meridian, a contradiction which establishes the claim. We conclude that if \( D_K(c) \) is innermost, then \( D_K(c) \) and \( D_T(c) \) bound a unique 3-ball \( E(c) \) in \( M \).

We now follow the procedure of Hatcher described in [17] to deform the family \( F \) to eliminate the circles of \( K_t \cap T_u \) that are inessential in \( T_u \). Since the method applies directly, we only give a sketch. By compactness, one can select a finite family of \( k \)-balls \( B_i \) covering \( D^k \), together with associated and distinct levels \( T_{u_i} \), so that for each \( t \in B_i \), each \( K_t \) is transverse to \( T_{u_i} \). Let \( C_i^t \) be the circles of \( K_t \cap T_{u_i} \), and \( C_t \) the union of the \( C_i^t \) over all \( t \) such that \( t \in B_i \). Smooth functions \( \varphi_t : C_t \to (0,1) \) are constructed satisfying \( \varphi_t(c_i) < \varphi_t(c'_i) \) whenever \( c_i, c'_i \in C_t \) and \( D_K(c_i) \subset D_K(c'_i) \) (for example, one can just take \( \varphi_t(c_i) \) to be the area of the disc \( F(t)^{-1}(D_K(c_i)) \) in \( K_0 \)). Now, a clever use of transversality, which may involve replacing a \( B_i \) with several smaller balls and the corresponding \( T_{u_i} \) by several nearby levels, enables one to assume that \( \varphi_t(c_i) \neq \varphi_t(c'_i) \) whenever \( c_i \neq c'_i \) in \( C_i^t \). The isotopy of \( F \) can then be constructed so that at each \( t \) it is a sequence of basic isotopies that push an innermost (at that point in the isotopy) \( D_K(c) \) across \( E(c) \) to eliminate \( c \) from \( K_t \cap T_u \). The \( \varphi_t \) give the order in which these basic isotopies are performed, ensuring that no two occur at the same time unless their 3-balls \( E(c) \) are disjoint. The isotopies may eliminate intersection circles scheduled for “later” removal, when they lie in \( D_T(c) \), but can never disturb the essential intersection circles. Moreover, none of the isotopies takes place at parameters in \( \partial D^k \), since no inessential intersection circles occur at these parameters. At the end of this process, for each \( t \in D^k \), there is a value \( u > 0 \) so that in place of (2) above we now have

\[(2') \quad \text{Every intersection circle of } K_t \text{ with } T_u \text{ represents either } a \text{ or } b^2 \text{ in } \pi_1(T_u) .\]

Since \( a \) and \( b^2 \) are nontrivial elements of \( \pi_1(M) \), the circles of \( K_t \cap T_u \) are essential in \( K_t \) as well, so each component of \( K_t \cap R_u \) must be either an annulus or a Möbius band. In fact, Möbius bands cannot occur. For the center circle of such a Möbius band would have intersection number 1 with \( K_t \) and intersection number 0 with \( K_0 \), contradicting the fact that \( K_t \) is isotopic to \( K_0 \).

Consider the annuli \( K_t \cap R_u \) (where \( u \) is one of the \( u_i \) for which \( t \in B_i \)) whose boundary circles are not isotopic in \( T_u \) to fibers. Notice that this does not occur at parameters in \( \partial D^k \). By condition (2') and lemma 5.1, any circle of \( K_t \cap R_u \) that is not isotopic in \( T_u \) to a fiber is also not a longitude of \( R_u \). So each such annulus \( A \) is parallel across a region \( W(A) \) in \( R_u \) to a uniquely determined annulus in \( T_u \). Also, note that any annuli \( A \) of \( K_t \cap R_u \) and \( A' \) of \( K_t \cap R_{u'} \) are either nested or disjoint. We again use the procedure of [17] to pull these annuli out of \( R_u \). In this case, \( C_i^t \) is the set of annuli of \( K_t \cap R_{u_i} \). The functions \( \varphi_t(A) \) can initially be defined to be the area of the annulus \( F(t)^{-1}(A) \) in \( K_0 \), then modified using the transversality method to
ensure that \( \varphi(A_i) \neq \varphi(A'_i) \) whenever \( A_i \neq A'_i \) in \( C_i \). The isotopies push the \( A \) across the \( W(A) \), possibly eliminating additional annuli of \( C_i \), and no isotopies take place at parameters of \( \partial D^k \). At the end of the process, \( K_t \) will actually be disjoint from \( R_u \) if the original intersection circles were not longitudes. In addition to (1) we now have

\[
(2'') \text{Every circle of } K_t \cap T_u \text{ is isotopic in } T_u \text{ to a fiber of the Seifert fibering on } M.
\]

To complete the argument, we require two technical lemmas.

**Lemma 7.1.** Let \( T \) be a torus with a fixed \( S^1 \)-fibering, and let \( C_n = \bigcup_{i=1}^n S_i \) be a union of \( n \) distinct fibers. Then \( \text{imb}(C_n, T) \to \text{imb}(C_n, T) \) is a weak homotopy equivalence. The same holds for the Klein bottle with either the meridional fibering or the longitudinal singular fibering.

**Proof.** First consider a surface \( F \) other than the 2-sphere, the disc, or the projective plane, with a base point \( x_0 \) in the interior of \( F \) and an embedding \( S^1 \subset F \) with \( x_0 \in S^1 \) which does not bound a disc in \( F \). In the next paragraph, we will sketch a argument using [15] that \( \text{imb}((S^1, x_0), (\text{int}(F), x_0)) \) has trivial homotopy groups. The approach is awkward and unnatural, but we have found no short, direct way to deduce this fact from [15] or other sources.

By the fibration theorem of Palais and Cerf [28], there is a fibration

\[
\text{Diff}(F \text{ rel } S^1) \cap \text{diff}(F, x_0) \to \text{diff}(F, x_0) \to \text{imb}((S^1, s_0), (\text{int}(F), x_0)).
\]

Since \( F \) is not the 2-sphere, disc, or projective plane, proposition 2 of [15] shows that \( \text{diff}(F, x_0) \) has the same homotopy groups as \( \text{diff}_1(F, x_0) \), the subgroup of diffeomorphisms that induce the identity on the tangent space at \( x_0 \), and by theorem 2 of [15], the latter is contractible. So we have isomorphisms

\[
\pi_{q+1}(\text{imb}((S^1, s_0), (\text{int}(F), x_0))) \cong \pi_q(\text{Diff}(F \text{ rel } S^1))
\]

for \( q \geq 1 \), and

\[
\pi_1(\text{imb}((S^1, s_0), (\text{int}(F), x_0))) \cong \pi_0(\text{Diff}(F \text{ rel } S^1) \cap \text{diff}(F, x_0)).
\]

Proposition 6 of [15] shows that the components of \( \text{Diff}(F \text{ rel } S^1) \) are contractible, so it remains only to see that only one component of \( \text{Diff}(F \text{ rel } S^1) \) is contained in \( \text{diff}(F, x_0) \). Suppose \( h \in \text{Diff}(F \text{ rel } S^1) \cap \text{diff}(F, x_0) \). If \( F' \) is the result of cutting \( F \) along \( S^1 \), then \( h \) induces \( h' \) on \( F' \). Notice that \( h' \) is the identity on \( \partial F' \), since \( h \) was orientation-preserving. Since \( h \) induces the identity on \( \pi_1(F, x_0) \), and \( S^1 \) does not bound a disc in \( F \), it follows that \( h' \) induces the identity on \( \pi_1(F', x'_0) \) at each of the copies \( x'_0 \) of \( x_0 \) in \( \partial F' \). So \( h' \) is homotopic to the identity of \( F' \), relative to \( \partial F' \). By lemma 1.4.2 of [40], \( h' \) is isotopic to the identity relative to \( \partial F' \). This shows that \( h \) was in the identity component of \( \text{Diff}(F \text{ rel } S^1) \).

We now start with the torus case of the lemma. Choose notation so that the \( S_i \) lie in cyclic order as one goes around \( T \), and fix basepoints \( s_i \) in \( S_i \).
for each \( i \). Consider the diagram
\[
\begin{array}{c}
\text{imb}_{f}(S_{n}, T \text{ rel } s_{n}) \rightarrow \text{imb}_{f}(S_{n}, T) \rightarrow \text{imb}(s_{n}, T) \\
\downarrow \quad \downarrow \quad \downarrow = \\
\text{imb}(S_{n}, T \text{ rel } s_{n}) \rightarrow \text{imb}(S_{n}, T) \rightarrow \text{imb}(s_{n}, T).
\end{array}
\]
The first row is a fibration by corollary 6.5 of [24] and the second by [28].

The fiber of the top row is homeomorphic to \( \text{Diff}_{+}(S_{n} \text{ rel } s_{n}) \), the group of orientation-preserving diffeomorphisms, which is contractible. We have already seen that the fiber of the second row is weakly contractible. Therefore the middle vertical arrow is a weak homotopy equivalence. For \( n = 1 \), this completes the proof, so we assume that \( n \geq 2 \).

Let \( A \) be the annulus that results from cutting \( T \) along \( S_{n} \), and let \( A_{0} \) be the interior of \( A \). Consider the diagram
\[
\begin{array}{c}
\text{imb}_{f}(C_{n-1}, A_{0} \text{ rel } S_{n-1}) \rightarrow \text{imb}_{f}(C_{n-1}, A_{0}) \rightarrow \text{imb}(s_{n-1}, A_{0}) \\
\downarrow \quad \downarrow \quad \downarrow = \\
\text{imb}(C_{n-1}, A_{0} \text{ rel } S_{n-1}) \rightarrow \text{imb}(C_{n-1}, A_{0}) \rightarrow \text{imb}(s_{n-1}, A_{0}).
\end{array}
\]

As before, the fibers are contractible, so the middle vertical arrow is a weak homotopy equivalence. Now consider the diagram
\[
\begin{array}{c}
\text{imb}_{f}(C_{n-1}, A_{0} \text{ rel } S_{n-1}) \rightarrow \text{imb}_{f}(C_{n-1}, A_{0}) \rightarrow \text{imb}_{f}(S_{n-1}, A_{0}) \\
\downarrow \quad \downarrow \quad \downarrow \\
\text{imb}(C_{n-1}, A_{0} \text{ rel } S_{n-1}) \rightarrow \text{imb}(C_{n-1}, A_{0}) \rightarrow \text{imb}(S_{n-1}, A_{0}).
\end{array}
\]
The first row is a fibration by corollary 6.5 of [24] and the second is a fibration by [28]. The right vertical arrow was shown to be a weak homotopy equivalence by the previous diagram. For \( n = 2 \), both fibers are points, so the middle vertical arrow is a weak homotopy equivalence. But \( \text{imb}_{f}(C_{n-1}, A_{0} \text{ rel } S_{n-1}) \) can be identified with \( \text{imb}_{f}(C_{n-2}, A_{0}) \), and similarly for the non-fiber-preserving spaces (cut \( A \) along \( S_{n-1} \) and take as the new \( A \) the component that contains \( C_{n-2} \)). So an induction on \( n \) shows that the middle vertical arrow is a weak homotopy equivalence for any value of \( n \).

To complete the proof, we use the diagram
\[
\begin{array}{c}
\text{imb}_{f}(C_{n}, T \text{ rel } S_{n}) \rightarrow \text{imb}_{f}(C_{n}, T) \rightarrow \text{imb}_{f}(S_{n}, T) \\
\downarrow \quad \downarrow \quad \downarrow \\
\text{imb}(C_{n}, T \text{ rel } S_{n}) \rightarrow \text{imb}(C_{n}, T) \rightarrow \text{imb}(S_{n}, T).
\end{array}
\]
The rows are fibrations, as in the first diagram of the proof. The right-hand vertical arrow is the case \( n = 1 \), already proven, and the map between fibers can be identified with \( \text{imb}_{f}(C_{n-1}, A_{0}) \rightarrow \text{imb}(C_{n-1}, A_{0}) \), which has been shown to be a weak homotopy equivalence for all \( n \).

For the Klein bottle case, the proof is line-by-line the same in the case of the meridinal fibering. For the longitudinal singular fibering, the only
difference is that rather than an annulus $A$, the first cut along $S_n$ produces a Möbius band.

**Lemma 7.2.** Let $\Sigma$ be a compact 3-manifold with nonempty boundary and having a fixed Seifert fibering. Let $F$ be a compact 2-manifold properly imbedded in $\Sigma$, such that $F$ is a union of fibers. Let $\text{imb}_{\partial f}(F, \Sigma)$ be the connected component of the inclusion in the space of (proper) imbeddings for which the image of $\partial F$ is a union of fibers. Then $\text{imb}_f(F, \Sigma) \rightarrow \text{imb}_{\partial f}(F, \Sigma)$ is a weak homotopy equivalence.

To prove lemma 7.2, we need a preliminary result.

**Lemma 7.3.** The following maps induced by restriction are fibrations.

(i) $\text{imb}(F, \Sigma) \rightarrow \text{imb}(\partial F, \partial \Sigma)$

(ii) $\text{imb}_{\partial f}(F, \Sigma) \rightarrow \text{imb}_f(\partial F, \partial \Sigma)$

(iii) $\text{imb}_f(F, \Sigma) \rightarrow \text{imb}_f(\partial F, \partial \Sigma)$.

**Proof.** Parts (i) and (iii) are cases of corollaries 9.3 and 9.4 of [24]. Part (ii) follows from part (i) since $\text{imb}_{\partial f}(F, \Sigma)$ is the preimage of $\text{imb}_f(\partial F, \partial \Sigma)$ under the fibration of part (i). □

**Proof of lemma 7.2.** First we use the following fibration from theorem 8.3 of [24],

$\text{Diff}(\Sigma \text{ rel } \partial \Sigma) \cap \text{diff}(\Sigma \text{ rel } \partial \Sigma) \rightarrow \text{diff}(\Sigma \text{ rel } \partial \Sigma) \rightarrow \text{diff}(\Sigma \text{ rel } \partial \Sigma)$

where $\Sigma$ is the quotient orbifold of $\Sigma$ and as usual $\text{Diff}_v$ indicates the diffeomorphisms that take each fiber to itself. The orbifold diffeomorphism group of $\Sigma$ can be identified with a subspace consisting of path components of the diffeomorphism group of the 2-manifold $B$ obtained by removing the cone points from $\Sigma$ (the subspace for which the permutation of punctures respects the orbit invariants associated to the corresponding cone points). Since $\partial B$ is nonempty, $\text{diff}(B \text{ rel } \partial B)$ and therefore $\text{diff}(\Sigma \text{ rel } \partial \Sigma)$ are contractible. Since $\pi_1(\text{diff}(\Sigma \text{ rel } \partial \Sigma))$ is trivial, the homotopy exact sequence of the fibration shows that $\text{Diff}_v(\Sigma \text{ rel } \partial \Sigma) \cap \text{diff}(\Sigma \text{ rel } \partial \Sigma)$ is connected, so is equal to $\text{diff}_v(\Sigma \text{ rel } \partial \Sigma)$. It is not difficult to see that each component of $\text{Diff}_v(\Sigma \text{ rel } \partial \Sigma)$ is contractible (see lemma 10.4 of [24] for a similar argument), so we conclude that $\text{diff}_f(\Sigma \text{ rel } \partial \Sigma)$ is weakly contractible.

Next, consider the diagram

$\text{Diff}_f(\Sigma \text{ rel } F \cup \partial \Sigma) \cap \text{diff}_f(\Sigma \text{ rel } \partial \Sigma) \rightarrow \text{diff}_f(\Sigma \text{ rel } \partial \Sigma) \rightarrow \text{imb}_f(F, \Sigma \text{ rel } \partial F)$

where the rows are fibrations by corollaries 8.7 and 3.6 of [24]. We have already shown that the components of $\text{Diff}_f(\Sigma \text{ rel } \partial \Sigma)$ and (by cutting along $F$) the components of $\text{Diff}_f(\Sigma \text{ rel } F \cup \partial \Sigma)$ are weakly contractible. By [16] (which, as noted in [16], extends to $\text{Diff}$ using [18]), the components of $\text{Diff}(\Sigma \text{ rel } \partial \Sigma)$ and $\text{Diff}(\Sigma \text{ rel } F \cup \partial \Sigma)$ are weakly contractible. Therefore to
show that \( \text{imb}_f(F, \Sigma \cup \partial F) \rightarrow \text{imb}(F, \Sigma \cup \partial F) \) is a weak homotopy equivalence it is sufficient to show that \( \pi_0(\text{Diff}_f(\Sigma \cup \partial F) \cap \text{Diff}(\Sigma \cup \partial F)) \rightarrow \pi_0(\text{Diff}(\Sigma \cup \partial F) \cap \text{Diff}(\Sigma \cup \partial F)) \) is bijective. It is surjective because every diffeomorphism of a Seifert-fibered 3-manifold which is fiber-preserving on the (non-empty) boundary is isotopic relative to the boundary to a fiber-preserving diffeomorphism (lemma VI.19 of [23]). It is injective because fiber-preserving diffeomorphisms that are isotopic are isotopic through fiber-preserving diffeomorphisms (see [41]).

The proof is completed by the following diagram in which the rows are fibrations by parts (iii) and (ii) of lemma 7.3, and we have verified that the left vertical arrow is a weak homotopy equivalence.

\[
\begin{array}{ccc}
\text{imb}_f(F, \Sigma \cup \partial F) & \longrightarrow & \text{imb}_f(F, \Sigma) \\
\downarrow & & \downarrow \\
\text{imb}(F, \Sigma \cup \partial F) & \longrightarrow & \text{imb}(\partial F, \partial \Sigma)
\end{array}
\]

\[
\begin{array}{ccc}
\text{imb}_f(F, \Sigma \cup \partial F) & \longrightarrow & \text{imb}_f(\partial F, \partial \Sigma) \\
\downarrow & & \downarrow \\
\text{imb}(F, \Sigma \cup \partial F) & \longrightarrow & \text{imb}(\partial F, \partial \Sigma)
\end{array}
\]

\(\square\)

We can now complete the proof of theorem 4.1 by deforming the family \( F \) to a fiber-preserving family. Since conditions (1) and (2) must remain true in a neighborhood of \( t \), we can cover \( D^k \) by convex \( k \)-cells \( B_j \), \( 1 \leq j \leq r \), having corresponding levels \( u_j \) for which (1) and (2) hold throughout \( B_j \). Since the transversality condition (1) is an open condition, we may slightly change the \( u \)-values, if necessary, to assume that the \( u_i \) are distinct. It is convenient to rename the \( B_j \) so that \( u_1 < u_2 < \cdots < u_r \).

Choose a PL triangulation \( \Delta \) of \( D^k \) sufficiently fine so that each \( i \)-cell lies in at least one of the \( B_j \). The deformation of \( F \) will take place sequentially over the \( i \)-skeleta of \( \Delta \). It will never be necessary to change \( F \) at points of \( \partial D^k \).

Suppose first that \( \tau \) is a 0-simplex of \( \Delta \). Let \( j_1 < j_2 < \cdots < j_k \) be the values of \( j \) for which \( \tau \subseteq B_j \). By condition (2), each intersection circle of \( K_\tau \) with each \( T_{j_q} \) is isotopic in \( T_{j_q} \) to a fiber of the Seifert fibering. We claim that they are also isotopic on \( K_\tau \) to an image of a fiber of \( K_0 \) under \( F(\tau) \). Since \( K_\tau \) is isotopic to \( K_0 \) and the intersection circles are 2-sided in \( K_\tau \), each intersection circle is isotopic in \( M \) to an \( a \)-loop or an \( b^2 \)-loop in \( K_0 \). When \( m = 1 \), \( b^2 \) is the generic fiber of \( M \), and \( a \) is not isotopic in \( M \) to \( b^2 \) since \( a = b^{2n} \) and \( n \neq 1 \). When \( n = 1 \), \( a \) is the fiber of \( M \), and \( b^2 \) is not isotopic to \( a \) since \( a^m = b^2 \) and \( m > 1 \). So the isotopy from \( K_\tau \) to \( K_0 \) carries the intersection loops to loops in \( K_0 \) representing the fiber. But \( a \)-loops are nonseparating and \( b^2 \)-loops are separating, so the loops must be isotopic in \( K_\tau \) to the image of the fiber of \( K_0 \) under \( F(\tau) \).

We may deform the parameterized family near \( \tau \), retaining transverse intersection with each \( T_{u_j} \) for which \( \tau \in B_j \), so that the intersection circles with these \( T_{u_j} \) are fibers and images of fibers. To accomplish this, first change \( F(\tau) \) by an ambient isotopy of \( M \) that preserves \( u \)-levels and moves
the intersection circles onto fibers in the $T_{u_j}$. Now, consider the preimages of these circles in $K_0$. We have seen that there is an isotopy that moves them to be fibers, changing $F(\tau)$ by this isotopy (and tapering it off in a small neighborhood of $\tau$ in $D^k$) we may assume that the intersection circles are fibers of $K_\tau$ as well. Now, using lemma 7.2 successively on the solid torus $R_{u_s}$, the product regions $R_{u_j} - R_{u_{j-1}}$ for $j = j_s, j_{s-1}, \ldots, j_2$, and the twisted I-bundle $P_{u_j}$, deform $F(\tau)$ to be fiber-preserving. These isotopies preserve the $u_j$-levels for which $\tau \in B_j$, so may be tapered off near $\tau$ so as not to alter any other transversality conditions.

Inductively, suppose that $F(t)$ is fiber-preserving for each $t$ lying in any $i$-simplex of $\Delta$. Let $\tau$ be an $(i+1)$-simplex of $\Delta$. For each $t \in \partial \tau$, $F(t)$ is fiber-preserving. Consider a $U_{t_j}$ for which $\tau \subset B_j$. The restriction of $F|_{\tau}$ to the preimage of $U_{t_j}$ is a parameterized family of imbeddings of a family of circles into $U_{t_j}$, which imbeds to fibers at each point of $\partial \tau$. By lemma 7.1, there is a deformation of $F|_{\tau}$, relative to $\partial \tau$, which makes $K_t \cap U_{t_j}$ consist of fibers in $U_{t_j}$ for each $t \in \tau$. We may change $F$ so as to move image points only very near $U_{t_j}$, and thereby not alter transversality with any other $U_{t_k}$. Now, the preimage of the intersection circles are a family of imbeddings of a collection of circles into $K_0$, which are fibers at points in $\partial \tau$, so using lemma 7.1 we may alter $F|_{\tau}$ so that the intersection circles are fibers of the $K_t$. Again using lemma 7.2 and proceeding from $R_{u_{j_s}}$ to $P_{u_{j_2}}$, deform $F$ on $\tau$, keeping it fixed over $\partial \tau$, to be fiber-preserving for all parameters in $\tau$. This completes the induction step and the proof of theorem 4.1.

References

1. (reference will be found)

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