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Homeomorphisms which are Dehn twists on the boundary

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Abstract A homeomorphism of a 3-manifold M is said to be Dehn twists on the boundary when its restriction to ∂M is isotopic to the identity on the complement of a collection of disjoint simple closed curves in ∂M. In this paper, we give various results about such collections of curves and the associated homeomorphisms. In particular, if M is compact, orientable, irreducible and ∂M is a single torus, and M admits a homeomorphism which is a nontrivial Dehn twist on ∂M, then M must be a solid torus.

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Introduction

A homeomorphism h of a compact 3-manifold M is said to be Dehn twists on the boundary when its restriction to ∂M is isotopic to the identity on the complement of a collection of disjoint simple closed curves in ∂M. If this collection is nonempty, and the restricted homeomorphism is not isotopic to the identity on the complement of any proper subset of the collection, then we say that h is Dehn twists *about* the collection. The restriction of h to ∂M is then isotopic to a composition of nontrivial Dehn twists about the curves, where for us a Dehn twist may be a power of a "single" Dehn twist. Note that the minimality condition implies that each curve in the collection is essential in ∂M, and no two of them are isotopic in ∂M. Our first main result gives strong restrictions on the collection of curves.

Theorem 1 *Let M be a compact orientable 3-manifold which admits a homeomorphism which is Dehn twists on the boundary about the collection $C_1, \ldots,$ C_n of simple closed curves in ∂M. Then for each i, either C_i bounds a disk in M, or for some $j \neq i$, C_i and C_j cobound an incompressible annulus in M.*

Our second main result gives structural information about such homeomorphisms. It refers to Dehn twists about disks and annuli in M, whose definition is recalled in Section 1 below.

Theorem 2 *Let M be a compact orientable 3-manifold which admits a homeomorphism which is Dehn twists on the boundary about the collection $C_1, \ldots,$ C_n of simple closed curves in ∂M. Then there exists a collection of disjoint imbedded disks and annuli in M, each of whose boundary circles is isotopic to one of the C_i, for which some composition of Dehn twists about these disks and annuli is isotopic to h on ∂M.*

That is, h must arise in the most obvious way, by composition of Dehn twists about a collection of disjoint annuli and disks with a homeomorphism that is the identity on the boundary.

Theorems 1 and 2 yield strong statements for specific classes of manifolds. For the case when M is a compression body, examined in Section 3 below, a homeomorphism which is Dehn twists on the boundary is actually isotopic to a product of Dehn twists about disjoint annuli and disks. This appears in [10] for the case when M is a handlebody. Another application is the following:

Corollary 3 *Let M be a compact orientable irreducible 3-manifold with boundary a torus. If M admits a homeomorphism which is a Dehn twist on ∂M, then M is a solid torus and the homeomorphism is isotopic to a Dehn twist about a meridian disk.*

In particular, the only knot complement in S^3 (indeed, the only irreducible complement of a knot in any closed orientable 3-manifold) admitting a homeomorphism which is a nontrivial Dehn twist on the boundary is the trivial knot.

Proof of Corollary 3 By Theorem 1, C bounds a disk in M. Since M is irreducible, this implies that M is a solid torus. By Theorem 2, there is a Dehn twist about the meridian disk which is isotopic on ∂M to the original homeomorphism, and since any homeomorphism which is the identity on the boundary of a solid torus is isotopic to the identity, this Dehn twist and the original homeomorphism must be isotopic. □

It appears that most of our results could be extended to the nonorientable case, adding the possibility of Dehn twists about Möbius bands in Theorems 1 and 2, and in Corollary 6, but the proof of Lemma 5 would require the more elaborate machinery of uniform homeomorphisms, found in [7] or chapter 12 of [2] (in particular, Lemma 12.1.2 of [2] is a version of Lemma 1.4 of [7] that applies to nonorientable 3-manifolds). Corollary 3 fails in the nonorientable case, however. Not only can a nonorientable manifold with torus boundary admit Dehn twists about Möbius bands, but an annulus can meet the torus boundary in such a way that a Dehn twist about the annulus will be isotopic on the boundary torus to an even power of a simple Dehn twist about one of its boundary circles.

Some of the work presented here is applied in the article "Knot adjacency, genus and essential tori," by E. Kalfagianni and X.-S. Lin [5]. We are grateful to the authors of that paper for originally bringing the possibility of results like Theorems 1 and 2 to our attention.

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1 Proof of Theorem 1

Recall that Dehn twists in 3-manifolds can be defined as follows. Consider first a properly imbedded and 2-sided disk or annulus F in a 3-manifold M. Imbed $F \times [0,1]$ in M so that $(F \times [0,1]) \cap \partial M = \partial F \times [0,1]$ and $F \times \{0\} = F$. Let r_θ rotate F through an angle θ (that is, if F is a disk, rotate about the origin, and if it is an annulus $S^1 \times [0,1]$, rotate in the S^1-factor). Fixing some integer n, define $t\colon M \to M$ by $t(x) = x$ for $x \notin F \times [0,1]$ and $t(z,s) = (r_{2\pi n s}(z), s)$ if $(z,s) \in F \times [0,1]$. The restriction of t to ∂M is a Dehn twist about each circle of ∂F. Dehn twists are defined similarly when F is a 2-sphere or a two-sided projective plane, Möbius band, torus, or Klein bottle (for the case of tori, there are infinitely many nonisotopic choices of an S^1-factor to define r_θ). Since a properly imbedded closed surface in M is disjoint from the boundary, a Dehn twist about a closed surface is the identity on ∂M.

The proof of Theorem 1 will use the following result on Dehn twists about annuli in orientable 3-manifolds.

Lemma 4 *Let A_1 and A_2 be properly imbedded annuli in an orientable 3-manifold M, with common boundary consisting of the loops C' and C''. Let N' and N'' be disjoint closed regular neighborhoods in ∂M of C' and C'' respectively, and let t_i be Dehn twists about the A_i whose restrictions to ∂M are supported on $N' \cup N''$. If the restrictions of t_1 and t_2 to N' are isotopic relative to $\partial N'$, then their restrictions to N'' are isotopic relative to $\partial N''$.*

Consequently, if A is a properly imbedded annulus whose boundary circles are isotopic in ∂M (in particular, if they are contained in a torus boundary component of M), then any Dehn twist about A is isotopic to the identity on ∂M.

Proof The result is clear if the A_i have orientations so that their induced orientations on $C' \cup C''$ are equal, since then the imbeddings of $S^1 \times I \times [0,1]$ into M used to define the Dehn twists can be chosen to agree on $S^1 \times \partial I \times [0,1]$. So we assume that the oriented boundary of A_1 is $C' \cup C''$ and the oriented boundary of A_2 is $C' \cup (-C'')$.

By assumption, t_1 and t_2 restrict to the same Dehn twist near C'. Their effects near C'' differ in that after cutting along C'', the twisting of C'' occurs in opposite directions, but since M is orientable, they also differ in that this twisting is extended to collar neighborhoods on opposite sides of C'' (that is, the imbeddings of $S^1 \times \partial I \times [0,1]$ used to define the Dehn twists fall on the same side of C' but on opposite sides of C''). Each of these differences changes a Dehn twist about C'' to its inverse, so their combined effect is to give isotopic Dehn twists near C''.

The last remark of the lemma follows by taking $A_1 = A$ and A_2 to be an annulus with $\partial A_2 = \partial A_1$, with A_2 parallel into ∂M. All Dehn twists about A_2 are isotopic to the identity on ∂M, so the same is true for all Dehn twists about A_1. $\square$

We will also need a fact about homeomorphisms of reducible 3-manifolds, even in many of the cases when M itself is irreducible.

Lemma 5 *Let $W = P\#Q$ be a connected sum of compact orientable 3-manifolds, with P irreducible. Let S be the sum 2-sphere. Suppose that ∂P is nonempty, and that $g\colon W \to W$ is a homeomorphism which preserves a component of ∂P. Then there is a homeomorphism $j\colon W \to W$, which is the identity on ∂W, such that $jg(S) = S$.*

Proof Let $P_1\#\cdots\#P_r\#R_1\#\cdots\#R_s$ be a prime factorization of W, where each P_i is irreducible and each R_j is $S^2 \times S^1$. Let Σ be the result of removing from a 3-sphere the interiors of $r+2s$ disjoint 3-balls $B_1, \ldots, B_r$, D_1, E_1, $D_2, \ldots, E_s$. For $1 \le i \le r$, let P_i' be the result of removing the interior of a small open 3-ball B_i' from P_i, and regard W as obtained from Σ and the union of the P_i' by identifying each ∂B_i with $\partial B_i'$ and each ∂D_j with ∂E_j.

In [7] and in Section 12.1 of [2], certain *slide homeomorphisms* of W are constructed. These can be informally described as cutting W apart along a ∂B_i or ∂D_j, filling in one of the removed 3-balls to obtain a manifold Y, performing an isotopy that slides that ball around a loop in the interior of Y, removing the 3-ball, and gluing back together to obtain a homeomorphism of the original W. Slide homeomorphisms are assumed to be the identity on ∂W (this is ensured by requiring that the isotopy that slides the 3-ball around the loop in Y be the identity on ∂Y at all times). Lemma 1.4 of [7], essentially due to M. Scharlemann, says that if T is a collection of disjoint imbedded 2-spheres in the interior of W, then there is a composition j of slide homeomorphisms such that $j(T) \subset \Sigma$.

Since P is irreducible, we may choose notation so that $P = P_1$ and $S = \partial B_1$. Applying Lemma 1.4 of [7] with $T = g(S)$, we obtain j so that $jg(S) \subset \Sigma$. In particular, there is a component Z of $W - jg(S)$ whose closure contains P_1'. Since g is assumed to preserve a component of ∂P_1, the closure of Z must be $jg(P_1')$. Since P_1 is irreducible, $jg(S)$ must be isotopic to S in W, so changing j by isotopy we obtain $jg(S) = S$. $\square$

We can now prove Theorem 1. Let N_j be disjoint closed regular neighborhoods of the C_j in ∂M, and let F be the closure of $\partial M - \cup_j N_j$. By hypothesis, we may assume that h is the identity on F. Let M' be another copy of M, and identify F with its copy F' to form a manifold W with boundary a union of tori, each containing one C_j. Denote by T_j the one containing C_j. Let $g\colon W \to W$ be h on M and the identity map on M', so that on each T_j, g restricts to a nontrivial Dehn twist about C_j.

Fix any C_i, and for notational convenience call it C_1. If W is irreducible, put $W_1 = W$. Otherwise, write W as $W_1 \# W_2$ where W_1 is irreducible and $T_1 \subseteq \partial W_1$, and let S be the sum sphere. By Lemma 5, there is a homeomorphism j of W that is the identity on ∂W, such that $jg(S) = S$. Split W along S, fill in one of the resulting 2-sphere boundary components to obtain W_1, and extend jg to that ball. This produces a homeomorphism g_1 of W_1 that restricts on each boundary torus of W_1 to a nontrivial Dehn twist about one of the C_j.

Assume first that W_1 has compressible boundary. Since W_1 is irreducible, it is a solid torus with boundary T_1. The only nontrivial Dehn twists on T_1 that extend to W_1 are Dehn twists about a meridian circle, showing that C_1 bounds a disk in W_1, and hence a disk E in W. Since C_1 does not meet F, we may assume that E meets F transversely in a collection of disjoint circles. The intersection X' of E with M' has a mirror image X in M. Change E by replacing X' with X, producing a singular disk in M with boundary C_1. By the Loop Theorem, C_1 bounds an imbedded disk in M.

We call the argument in the previous paragraph that started with E in W and obtained a singular version of E in M, having the same boundary as the original E, a *swapping* argument (since we are swapping pieces of the surface on one side of F for pieces on the other side).

Suppose now that W_1 has incompressible boundary. Let V_1 be Johannson's characteristic submanifold of W_1 ([4], also see Chapter 2 of [2] for an exposition of Johannson's theory). Since ∂W_1 consists of tori, V_1 admits a Seifert fibering and contains all of ∂W_1 (in Johannson's definition, a component of V_1 can be just a collar neighborhood of a torus boundary component). Each C_j in W_1 is noncontractible in T_j, and T_j is incompressible in W_1, so C_j is noncontractible in W_1. This implies that C_j is noncontractible in W, hence also in M.

It suffices to prove that C_1 and some other C_i cobound an imbedded annulus A in W_1 and hence in W. For then, a swapping argument produces a singular annulus in M cobounded by C_1 and C_i. Since C_1 and C_i are noncontractible, a direct application of the Generalized Loop Theorem ([12], see p. 55 of [3]) produces an imbedded annulus in M cobounded by C_1 and C_i.

By Corollary 27.6 of [4], the mapping class group of W_1 contains a subgroup of finite index generated by Dehn twists about essential annuli and tori. So by raising g_1 to a power, we may assume that it is a composition of such Dehn twists. The Dehn twists about tori do not affect ∂W_1, so we may discard them to assume that g_1 is a composition $t_1 \cdots t_m$, where each t_k is a Dehn twist about an essential annulus A_k. By Corollary 10.10 of [4], each A_k is isotopic into V_1. By Proposition 5.6 of [4], we may further change each A_k by isotopy to be either horizontal or vertical with respect to the Seifert fibering of V_1.

Suppose first that some A_k is horizontal. Then V_1 is either $S^1 \times S^1 \times I$ or the twisted I-bundle over the Klein bottle (a horizontal annulus projects by an orbifold covering map to the base orbifold, and the orbifold Euler characteristic shows that the base orbifold is either an annulus, a Möbius band, or the disk with two order-2 cone points, the latter two possibilities yielding the two Seifert fiberings of the twisted I-bundle over the Klein bottle). In the latter case, $\partial V_1 = T_1$, so $W_1 = V_1$ and therefore $\partial W_1 = T_1$. By Lemma 4, each t_k is isotopic to the identity on T_1, hence so is g_1, a contradiction. So $V_1 = S^1 \times S^1 \times I$.

Since A_k is horizontal, it must meet both components of ∂V_1, and we have $V_1 = W_1$ and $\partial W_1 = T_1 \cup T_i$ for some i. Let $A_0 = C_1 \times I \subset S^1 \times S^1 \times I$. For an appropriate Dehn twist t about A_0, $t^{-1}g$ is isotopic to the identity on T_1. Using Lemma 3.5 of [11], $t^{-1}g$ is isotopic to a level-preserving homeomorphism of W_1, and hence to the identity. We conclude that g_1 is isotopic to t, and consequently C_1 and C_i cobound an annulus in W_1.

It remains to consider the case when all A_k are vertical. In this case, each t_k restricts on ∂W_1 to Dehn twists about loops isotopic to fibers, so each C_j in ∂W_1 is isotopic to a fiber of the Seifert fibering on V_1.

Let V_1' be the component of V_1 that contains C_1. Suppose first that $V_1' \cap \partial W_1 = T_1$. Then each A_k that meets T_1 has both boundary circles in T_1, so Lemma 4 implies that g_1 is isotopic to the identity on T_1, a contradiction. So V_1' contains another T_i. Since C_1 and C_i are isotopic to fibers, there is an annulus in V_1' with boundary $C_1 \cup C_i$.

2 Proof of Theorem 2

Theorem 1 provides a properly imbedded surface S which is either an imbedded disk with boundary C_n or an incompressible annulus with boundary C_n and some other C_i. For some Dehn twist t_n about S, t_n and h are isotopic near C_n. The composition $t_n^{-1}h$ is isotopic on ∂M to a composition of Dehn twists about $C_1, \ldots, C_{n-1}$ (some of them possibly trivial). Induction on n produces a composition t as in the theorem, except for the assertion that the disks and annuli may be selected to be disjoint.

Let $D_1, \ldots, D_r$ and $A_1, \ldots, A_s$ be the disks and annuli needed for the Dehn twists in t. We first work on the annuli.

We will say that a union $\mathcal{A}$ of disjoint incompressible imbedded annuli in M is *sufficient* for $A_1, \ldots, A_k$ if each boundary circle of $\mathcal{A}$ is isotopic in ∂M to a boundary circle of one of the A_i, and if for any composition of Dehn twists about $(\cup_{i=1}^r D_i) \cup (\cup_{j=1}^k A_j)$, there is a composition of Dehn twists about the union of $\cup_{i=1}^r D_i$ and the annuli of $\mathcal{A}$ which has the same effect, up to isotopy, on ∂M. In particular, $\mathcal{A} = A_1$ is sufficient for A_1 alone. Inductively, suppose that $\mathcal{A}$ is sufficient for $A_1, \ldots, A_{k-1}$. By a routine surgery process, we may change $\mathcal{A}$ so that A_k and $\mathcal{A}$ intersect only in circles essential in both A_k and $\mathcal{A}$. (First, make $\mathcal{A}$ transverse to A_k. An intersection circle which is contractible in $\mathcal{A}$ must also be contractible in A_k, since both $\mathcal{A}$ and A_k are incompressible. If there is a contractible intersection circle, then there is a disk E in A_k with ∂E a component of $A_k \cap \mathcal{A}$ and the interior of E disjoint from $\mathcal{A}$. Replace the disk in $\mathcal{A}$ bounded by ∂E with E, and push off by isotopy to achieve a reduction of $A_k \cap \mathcal{A}$.)

Now let Z be a closed regular neighborhood of $A_k \cup \mathcal{A}$. Since all intersection circles of A_k with $\mathcal{A}$ are essential in both intersecting annuli, each component of Z has a structure as an S^1-bundle in which the boundary circles of $\mathcal{A}$ and A_k are fibers.

We will show that Z contains a collection sufficient for $A_k \cup \mathcal{A}$ and hence for $A_1, \ldots, A_k$. We may assume that Z is connected. For notational simplicity, there is no harm in writing $C_1, \ldots, C_m$ for the boundary circles of $\mathcal{A}$ and A_k, since they are isotopic in ∂M to some of the original C_i.

Fix a small annular neighborhood N of C_1 in $Z \cap \partial M$. Using the S^1-bundle structure of Z, we can choose a collection $B_2, \ldots, B_m$ of disjoint annuli, with B_i running from C_i to a loop in N parallel to C_1.

Consider one of the annuli A of $A_k \cup \mathcal{A}$, say with boundary circles isotopic to C_i and C_j. If either i or j is 1, say $j = 1$, then by Lemma 4, Dehn twists about A have the same effect on ∂M as Dehn twists about B_i. If neither is 1, form an annulus B connecting C_i to C_j by taking the union of B_i, B_j, and the annulus in N connecting $B_i \cap N$ to $B_j \cap N$, then pushing off of N to obtain a properly imbedded annulus. Observe that any Dehn twist about B is isotopic on M to a composition of Dehn twists about B_i and B_j. By Lemma 4, there is a Dehn twist about B whose effect on ∂M is the same as the twist about A. This shows that the collection $B_2, \ldots, B_m$ is sufficient for $A_1, \ldots, A_k$ and completes the induction. So there is a collection $\mathcal{A}$ sufficient for $A_1, \ldots, A_s$.

By further routine surgery, we may assume that each D_i is disjoint from $\mathcal{A}$. Then, surger D_2 to make D_2 disjoint from D_1, surger D_3 to make it disjoint from $D_1 \cup D_2$, and so on, eventually achieving the desired collection of disjoint disks and annuli.

3 Compression bodies

Compression bodies were developed by F. Bonahon [1], in a study of cobordism of surface homeomorphisms. They were used in work on mapping class groups of 3-manifolds in [9], [6], and [8], and on deformations of hyperbolic structures on 3-manifolds in [2]. The homeomorphisms of compression bodies were further investigated in [10], which develops an analogue for compression bodies of the Nielsen-Thurston theory of surface homeomorphisms.

To fix notation and terminology, we recall that a *compression body* is a connected 3-manifold V constructed by starting with a compact surface G with no components that are 2-spheres, forming $G \times [0,1]$, and then attaching 1-handles to $G \times \{1\}$. Compression bodies are irreducible. They can be handlebodies (when no component of G is closed) or product I-bundles (when there are no 1-handles). The *exterior boundary* of V is $\partial V - (G \times \{0\} \cup \partial G \times [0,1))$. Note that if F is the exterior boundary of V, and N is a (small) regular neighborhood in V of the union of F with a collection of cocore 2-disks for the 1-handles of V, then each component of $\overline{V-N}$ is a product $X \times I$, where $X \times \{0\}$ is a component of the frontier of N and $X \times \{1\}$ is a component of $G \times \{0\}$.

The following result was proven in [10] for the case of V a handlebody.

Corollary 6 *Let V be a compact orientable compression body, and let $h\colon V \to V$ be a homeomorphism which is Dehn twists on the boundary about the collection $C_1, \ldots, C_n$ of simple closed curves in ∂V. Then h is isotopic to a composition of Dehn twists about a collection of disjoint disks and incompressible annuli in V, each of whose boundary circles is isotopic in ∂V to one of the C_i.*

To prove Corollary 6, we note first that by Theorem 2, there is a composition t of Dehn twists about a collection of disjoint disks and incompressible annuli in V, such that t and h are isotopic on ∂V. Changing h by isotopy, we may assume that $t^{-1}h$ is the identity on ∂V. Corollary 6 is then immediate from the following lemma.

Lemma 7 *Let V be a compression body with exterior boundary F, and let $g\colon V \to V$ be a homeomorphism which is the identity on F. Then g is isotopic relative to F to the identity.*

Proof We have noted that there is a collection of disjoint properly imbedded disks $E_1, \ldots, E_n$, with boundaries in F, such that if N is a regular neighborhood of $F \cup (\cup_i E_i)$, then each component of $\overline{V-N}$ is a product $X \times I$, where $X \times \{0\}$ is a component of the frontier of N. Now ∂E_1 is fixed by g, so we may assume that $g(E_1) \cap E_1$ consists of ∂E_1 and a collection of transverse intersection circles. Since V is irreducible, we may change g by isotopy relative to F to eliminate these other intersection circles, and finally to make g fix E_1 as well as F. Inductively, we may assume that g is the identity on $F \cup (\cup_i E_i)$, and then on N. Finally, for each component $X \times I$ of $\overline{V-N}$, g is the identity on $X \times \{0\}$. Using Lemma 3.5 of [11], g may be assumed to preserve the levels $X \times \{s\}$ of $X \times I$, and then there is an obvious isotopy from g to the identity on $X \times I$, relative to $X \times \{0\}$. Applying these isotopies on the complementary components of N, we make g the identity on V. $\square$

References

[1] **F Bonahon**, *Cobordism of automorphisms of surfaces,* Ann. Sci. École Norm. Sup. (4) 16 (1983), 237–270.

[2] **R Canary**, **D McCullough**, *Homotopy equivalences of 3-manifolds and deformation theory of Kleinian groups,* Mem. Amer. Math. Soc. 172 (2004) no. 812, 1–218.

[3] **J Hempel**, 3*-manifolds*, Princeton University Press, 1976.

[4] **K Johannson**, *Homotopy Equivalences of* 3*-manifolds with Boundary*, Lecture Notes in Mathematics, vol. 761, Springer-Verlag, 1979.

[5] **E Kalfagianni**, **X.-S. Lin**, *Knot adjacency, genus and essential tori,* to appear in Pacific J. Math.

[6] **D McCullough**, *Twist groups of compact* 3*-manifolds,* Topology 24 (1985), 461–474.

[7] **D McCullough**, *Topological and algebraic automorphisms of* 3*-manifolds,* from: "Groups of Self-Equivalences and Related Topics," Lecture Notes in Mathematics, vol. 1425, Springer-Verlag (1990), 102–113.

[8] **D McCullough**, *Virtually geometrically finite mapping class groups of* 3*-manifolds,* J. Diff. Geom. 33 (1991), 1–65.

[9] **D McCullough**, **A Miller**, *Homeomorphisms of* 3*-manifolds with compressible boundary,* Mem. Amer. Math. Soc. vol. 61 no. 344 (1986).

[10] **U Oertel**, *Automorphisms of* 3*-dimensional handlebodies,* Topology 41 (2002), 363–410.

[11] **F Waldhausen**, *On irreducible* 3*-manifolds which are sufficiently large,* Annals of Math. 87 (1968), 56–88.

[12] **F Waldhausen**, *Eine Verallgemeinerung des Schleifensatzes,* Topology 6 (1967), 501–504.

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