Ends of Cusp-uniform Groups of locally connected Continua – I

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Abstract

Due to works by Bestvina-Mess, Swarup and Bowditch, we now have complete knowledge of how splittings of a word-hyperbolic group $G$ as a graph of groups with finite or two-ended edge groups relate to the cut point structure of its boundary. It is central in the theory that $\partial G$ is a locally connected continuum (a Peano space). Motivated by the structure of tight circle packings, we propose to generalize this theory to cusp-uniform groups in the sense of Tukia.

A Peano space $X$ is cut-rigid, if $X$ has no cut point, no points of infinite valence and no cut-pairs consisting of bivalent points. We prove:

**Theorem:** Suppose $X$ is a cut-rigid space admitting a cusp-uniform action by an infinite group. If $X$ contains a minimal cut triple of bivalent points, then there exists a simplicial tree $T$, canonically associated with $X$, and a canonical simplicial action of $\text{Homeo}(X)$ on $T$ such that any infinite cusp-uniform group $G$ of $X$ acts cofinitely on $T$, with finite edge stabilizers.

In particular, if $X$ is such that $T$ is locally finite, then any cusp-uniform group $G$ of $X$ is virtually free.

1 Introduction

Suppose $G$ is a group acting minimally on a locally connected metrizable compactum $X$ (a Peano space) as a (non-elementary) cusp-uniform convergence group in the sense of Tukia ([16]), that is: every point of $X$ is either a conical limit point or a bounded parabolic point (see 2.1 below). In this
paper, we relate the number of ends of $G$ to combinatorial properties of finite cut sets in $X$.

**Word-hyperbolic groups – a motivating example.**

One major motivating class of examples for this study and the constructions herein is given by actions $G \rtimes \partial G$, where $G$ is a word-hyperbolic group and $X = \partial G$ is its canonical boundary. It is known that $G$ is one-ended if and only if $\partial G$ is connected, meaning $G$ cannot split (as an amalgam or an HNN extension) over a finite group. It is a deep result of Bestvina-Mess ([1]), Swarup ([14]) and Bowditch ([4]), that $\partial G$ is locally connected and has no global cut point – a point whose complement in $\partial G$ is disconnected. Finally, if $\partial G$ is connected, it was shown that any splitting of $G$ over a virtually-cyclic group $C$, corresponds to a certain cut-pair in $\partial G$ stabilized by $C$ ([2]).

This last result may be thought of as an analog of what is known in the theory of Kleinian groups as a *combination theorem*, though it is better thought of as a *decomposition* theorem. There, one considers a family of geometrically-finite Kleinian groups $G_i \subset \text{Isom}(\mathbb{H}^n)$ and asks when the group $G$ generated by their union arises as a graph of groups with the $G_i$ being, say, vertex groups for this graph (here the counterpart of $\partial G$ is the limit set of the group $G$).

In comparison, Bowditch’s result ([2]) gives a way to “reverse engineer” the groups $G_i$ from the group $G$, provided one has enough information about the way cut-pairs are arranged in $\partial G$. It is the special kind of dynamics $G$ has on $\partial G$ that allows one to associate cut-pairs to splittings of $G$ over two-ended subgroups.

This work arose from an ambition to apply Bowditch’s ideas (“the dynamic approach”) to an arbitrary geometrically finite Kleinian group $G$ having a locally connected limit set $X = \Lambda(G)$ in order to reveal the “combinations” from which it may arise. Progress in this direction might shed more light on the old problem of effectively characterizing geometrically finite groups with locally connected limit sets.

The difficulty with the dynamic approach to Kleinian groups lies in that one seems to discard all the geometry of this setting. For example, working with the bare topology of $X$ seems to leave all information regarding the way $X$ is embedded in the sphere at infinity completely ignored.

Still, it is precisely this way of perceiving a Kleinian group $G$ that allows one to find the right setting accommodating both the class of all word-hyperbolic groups (acting on their boundaries) and the class of all geometrically finite Kleinian groups (acting on their limit sets). It is known that a group $G$ is
word-hyperbolic if and only if ([11, 3]) $G$ acts \textit{uniformly} on a metrizable compactum $X$, in which case $X$ is equivariantly homeomorphic to the Gromov boundary of $G$.

A more general notion is that of a \textit{discrete convergence group action} introduced by Gehring and Martin in [12] as a generalization of discrete groups of conformal mappings of the sphere, and later extended to arbitrary metrizable spaces by Tukia. We refer the reader to 2.1 for the definitions regarding convergence groups.

It so happens that both cases of interest fit into the class of \textit{cusp-uniform} convergence groups defined by Tukia [16] and subsequently investigated by Yaman [18] and Dahmani [9]. In [16] it is shown that a Kleinian group $G$ is geometrically finite if and only if its action on the limit set $\Lambda(G)$ is cusp-uniform. Therefore, for our purposes the class of cusp-uniform groups seems to be a “best fit”.

\textbf{Topological differences.}

One fundamental difference between a uniform action $G \curvearrowright X$ and a cusp-uniform one is that, in the former, $X$ is uniquely determined up to equivariant homeomorphism by the group $G$, which is necessarily a word-hyperbolic group ([3]). A similar result exists for cusp-uniform groups ([18]), but $G$ is not enough to determine $X$: one needs to fix a family of so-called \textit{parabolic subgroups} in order for $X$ to become well-defined. Basically, this is the reason why it is easy to find geometrically finite (but not uniform) actions of a word-hyperbolic group $G$ – say, a free group – on Peano continua (see discussion of circle packings below, 2.2): while a free group may not act uniformly on anything but a Cantor set, it may act cusp-uniformly on continua like the Apollonian packing. Thus, one important question regarding the correspondence between cut sets and splittings is:

\textbf{Question:} Suppose a group $G$ acts cusp-uniformly on a Peano continuum $X$. Does the cut point structure of a space $X$ provide information about the splittings of $G$ over finite subgroups?

The technique developed in [2] may roughly be described as follows. Assuming $X$ is a locally connected continuum without a global cut point, if $\mathcal{F}$ is a family of \textit{disjoint} finite cut sets invariant under homeomorphisms of $X$ and such that every pair of elements $A, B \in \mathcal{F}$ are, in some sense, nested, then one may impose on $\mathcal{F}$ an invariant structure of a pretree. If this pretree is discrete, then \textit{any} group $G$ acting on $X$ acts on an “enveloping” simplicial tree $T$, producing splittings whenever $G$ has no fixed point in $T$. To this end, it is quite obvious that in the case of a one-ended word-hyperbolic group
acting on its boundary one should take $\mathcal{F}$ to be a family of cut pairs: hopefully, the stabilizer of a cut pair is infinite, and then it has to be two-ended (this follows from general facts about convergence groups), and we obtain a splitting over a two-ended group.

In [2], it is shown that if a Peano continuum $X$ admits a uniform action by a group $G$, then all points of valence greater than 2 are paired into fixed point sets of loxodromic elements of $G$, and that these pairs constitute a nested family; the points of valence 2 are, too, either paired in the same manner, or divided into cyclically-separating families. Bowditch then uses a variation of the usual notion of separation for disjoint cut sets to define a notion of separation on the set of all loxodromic cut pairs and all cyclically-separating families of bivalent points.

For the case of a general convergence group action one has to check the number of ends first. We proceed by analogy: since $G$ acts properly-discontinuously on $3X$, if $A$ is a cut set containing more than 3 points, then its stabilizer is finite, and again, hopes are that one gets a a splitting of $G$ over $G_A$ either through taking $\mathcal{F}$ to be the orbit of $A$ or – even better, since the goal is to arrive at a constructive answer in the spirit of [2], – through taking $\mathcal{F}$ to be, say, the family of all cut sets in $X$ of size $|A|$.

The example of a geometrically finite group $G$ of isometries of hyperbolic space acting minimally on a circle packing $X$ becomes an extremely discouraging example in this context (see 2.2 for an overview): we note that all the local cut points in $X$ are parabolic points of $G$, so that none of the results of [2] apply; the notion of separation generally used for continua fails to cover all possible configurations of finite cut sets arising in $X$ – in particular, one has to consider a notion of separation for intersecting cut sets; the orbit of a single cut set of size greater than 3 is seldom nested; special configurations of cut sets of different sizes may result in a seemingly hopeless situation regarding discreteness properties.

This set of examples necessitated the development of the new generalized notion of separation presented in this paper, as well as the introduction of technical tools to use it. All this is presented in section 3, and applied directly to our setting in section 4.

**Results.**

It turns out, due to the uniform approximation theorem of Tukia ([16], theorem 1C), that our setting ($X$ – a locally connected continuum admitting a cusp-uniform action by a group $G$) retains many of the important properties of the analogous uniform setting. For example, we prove a global bound on
the valence of conical limit points in $X$ and we characterize multivalent points as loxodromic fixed points of any cusp-uniform group $G$ of $X$. We present these in section 4. These results show that every multivalent point belongs to a unique “symmetric” cut pair in $X$, which is the basic phenomenon that allowed Bowditch to construct his tree.

As for bivalent points, then the crucial observation turned out to be that, disallowing points of infinite valence in $X$, one can bound the valence of a parabolic non-cut point by the number 2 (see proposition 4.4).

Despite this optimistic observation, we have noted already that the future of “treelike” analysis of the structure of the system of finite cuts seems pretty bleak due to its extreme complexity (as opposed to what happens in the uniform case).

We partially overcome this difficulty using a notion of symmetry (we call it exactness) that may be applied to finite cuts, generalizing the symmetry one observes in loxodromic cut-pairs (the type of cuts responsible for the cut point structure in boundaries of word-hyperbolic groups). It seems plausible that the cut sets playing a major role in the structure of a group $G$ acting on a space $X$ should be, in some sense, symmetric, and it came as a surprise that merely the assumption that $X$ carries some cusp-uniform action by an infinite group imposes on the family of finite exact cuts very powerful combinatorial restrictions allowing to import ideas from circle packings into the general setting of a Peano space carrying a cusp-uniform group action.

In this work, the main actors become the exact cut triples, which are shown – on one hand – to form the only natural family of large minimal cuts having a treelike nesting structure, and – on the other hand – play in $X$ the same role as that of Jordan curve on the 2-sphere.

All these are discussed in section 4. The reader looking for examples motivating the constructions therein should find our discussion of circle packings in paragraph 2.2 helpful.

We define a topological generalization of circle-packings: call a Peano space $X$ cut-rigid, if $X$ contains no global cut point, no points of infinite valence and no bivalent exact cut-pairs; our main theorem then is –

**Theorem A:** Suppose $X$ is a cut-rigid space admitting a cusp-uniform action by an infinite group. If $X$ contains a minimal cut triple of bivalent points, then there exists a simplicial tree $T$, canonically associated with $X$, and a canonical simplicial action of Homeo($X$) on $T$ such that any infinite cusp-uniform group $G$ of $X$ acts cofinitely on $T$, with finite edge stabilizers.
We prove this theorem by showing that the (topologically invariant) set of minimal \textit{exact} (“symmetric”) cut-triples has the structure of a discrete pretree (this is done in section 5, where Theorem A is proved in its most general form). We then use an idea of Bowditch to embed this pretree in a simplicial tree $\Xi_3$, in which the cut triples correspond to midpoints of edges (the particular embedding used is described in paragraph 2.3).

We also show vertex stabilizers are controlled by the topology of $X$: for example, the stabilizer of a vertex is finite iff the vertex has finite valence in $\Xi_3$. Thus, if $X$ has a locally-finite $\Xi_3$, it is as close to being a circle packing of finite type as one could hope for: every cusp-uniform group of $X$ is finitely generated virtually free, being the fundamental group of a finite graph of finite groups.

Completely classifying the vertex stabilizers of $\Xi_3$ under a cusp-uniform group action turned out to be a hard, though interesting, task. There are examples of infinite stabilizers known to the author – but always virtually-cyclic ones. Therefore, it might be tempting to conjecture that any $G$ such as ours has to be finitely-generated virtually free. This particular point seems to be relevant to an alternative approach to proving $G$ has more than one end, which was proposed by a referee and is discussed in section 6. A more ambitious goal would be to obtain some sort of converse to theorem A. We formulate some questions in the closing section of the paper.

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2 Preliminaries

2.1 discrete convergence groups

Let a group $G$ act by homeomorphisms on a compact metrizable space $X$. It will be convenient for us to denote by $3X$ the space of all ordered triples of distinct elements of $X$.

The following definition is actually a characterization rather than the original definition:

**Definition 2.1 (convergence group – compare [12, 5, 16])** A group $G$ acting by homeomorphisms on a compact space $X$ is said to be a discrete convergence group, if $G$ acts properly discontinuously on $3X$.

For an arbitrary convergence group we have the following classification of elements:

**Definition 2.2 (see [12, 5])** Suppose $G$ is a convergence group of a metrizable compact space $X$. Then an element $g \in G$ is said to be

- elliptic, if $g$ has finite order;
- loxodromic, if $g$ has infinite order and precisely two distinct fixed points in $X$;
- parabolic, if $g$ has infinite order and precisely one fixed point in $X$.

It is known (see references above) that every element of $G$ is of one of the above types.

**Definition 2.3 (conical point, parabolic point – see [16])** Suppose $G$ is a convergence group of a metrizable compact space $X$. A point $p \in X$ is conical if there are a sequence $(g_n \in G)_{n=1}^{\infty}$ and distinct points $a, b \in X$ such that $g_np \to a$, while $g_ny \to b$ for all $y \neq p$ in $X$; $p$ is parabolic if the stabilizer $G_p$ of $p$ is infinite and contains no loxodromic element; A parabolic point $p$ is bounded if $G_p$ acts co-compactly on $X \setminus \{p\}$. The groups $G_p$ are the maximal parabolic subgroups of $G$.

Tukia proved in [16] (there, Theorem 1A) that *all* the points of $X$ are conical limit points of $G$ if and only if, in addition to its proper discontinuity, the diagonal action of $G$ on $3X$ is co-compact. In this case, the group $G$ is said to act *uniformly* on the space $X$, and Bowditch, in [4], shows that a group $G$ is
word-hyperbolic if and only if it acts as a uniform group on some metrizable compact space.

For the more general case, Tukia, in [16], devised a notion generalizing uniform actions in the sense that the quotient of $3X$ by $G$ need not be compact, but has to be compact up to the addition of finitely many cusps. In the same paper, Tukia characterized this situation as follows (here we use the characterization as a definition):

**Definition 2.4 (see [16], theorem 1B)** A convergence group $G$ of a space $X$ is said to be cusp-uniform, if every non-isolated point of $X$ is either a conical limit point, or a bounded parabolic point.

It is a deep result of the same work, that a cusp-uniform group has only $G$-finitely many (bounded) parabolic points. Tukia’s proof is based on regarding $X$ as the boundary of a compactification of $3X$ as follows: for any point $x \in X$ and any neighbourhood $U$ of $x$, set $\hat{U}$ to be the set of all triples $t \in 3X$ having at-least two coordinates inside $U$; we then let the topology on $3X \coprod X$ be the one generated by the topology of $3X$ and all the sets $\tilde{U}$ defined as above. Next, for any pair of distinct points $x, x' \in X$ Tukia defined the line $L(x, x')$ with initial point $x$ and endpoint $x'$ to be the set of all triples in $3X$ of the form $(x, x', y)$. This enabled a characterization of conical limit points as “radial limit points” in the following sense:

**Lemma 2.5 (see [16], section 1C)** A point $x \in X$ is a conical limit point of a convergence group $G$ if and only if for some (and hence any) line $L = L(x, x')$ in $3X$ there exists a compact $K \subset 3X$ such that $L \cap GK$ accumulates at $x$.

We shall also require the result on uniform approximability of conical limit points of a cusp-uniform group from [16]:

**Theorem 2.6 (see [16], theorem 1C, part (a))** Let $G$ be a cusp-uniform group of $X$. Then there exists a compact set $K \subset 3X$ such that if $x \in X$ is not isolated and not a bounded parabolic point then for every line $L$ with initial point $x$ we have $L \cap GK$ accumulates at $x$.

### 2.2 Example: Circle Packings and their symmetries.

Looking for cusp-uniform group actions on Peano spaces, one finds out that tight planar circle packings are the first reasonable (and rewarding) objects to study in this context.
Let us now fix some notation and recall necessary facts. Let $\Gamma$ be the image of an embedding of a finite connected planar graph in the $2$-sphere $S^2$ such that there exists a collection of circles $P^0(\Gamma) = \{C_v\}_{v \in V\Gamma}$ satisfying

1. For all $v \in V$, $C_v$ bounds a disk not intersecting any of the circles in $P^0(\Gamma)$;

2. $C_u$ and $C_v$ are tangent if and only if the edge $\{u, v\}$ lies in $E\Gamma$, and are otherwise disjoint.

Such a collection of circles is called a (finite) circle packing.

Now, for every region $R$ of the complement of $\Gamma$, let $\Delta(R)$ denote the set of vertices corresponding to the indices of the circular arcs bounding $R$, and, whenever possible, let $C'_R$ be the unique circle through all the tangency points (of $P^0(\Gamma)$) belonging to the boundary of $R$. For example, $C'_R$ is defined whenever $\Delta(R)$ is a triangle in $\Gamma$.

Let now $I_R$ be the inversion in the circle $C'_R$. We see that the union $P^0(\Gamma) \cup I_R(P^0(\Gamma))$ is again a packing if and only if $C'_R$ is orthogonal to all the circles $C_v$ it intersects. We let $I(\Gamma)$ be the set of all inversions arising in this way, and denote by $G(\Gamma)$ the group generated by $I(\Gamma)$. Finally, let $P(\Gamma)$ be the closure of the orbit of $\bigcup P^0(\Gamma)$ under the action of $G(\Gamma)$, and refer to it as the tight circle-packing of type $\Gamma$.

Now, suppose $\Gamma$ is such that the packing $P^0(\Gamma)$ is contained in another finite packing $P$ with the property that every region $R$ of $S^2 \setminus \bigcup P$ is triangular. In this case it is known that the group $G(\Gamma)$ has a finite-sided fundamental domain (when viewed as a groups of isometries of hyperbolic space), and it is easily deduced that $P(\Gamma)$ is the limit set of $G(\Gamma)$, so that $G(\Gamma)$ is a cusp-uniform group with boundary $X = P(\Gamma)$ (see, for instance, [7, 6]). One famous example of such a packing is that of the Apollonian packing arising for the complete graph on $4$ vertices (see 3.2). We shall refer to the spaces $P(\Gamma)$ as circle packings of finite type.

Other packings, too, arise as limit sets of geometrically-finite groups of hyperbolic isometries, but we leave the analysis of the subtle differences between such packings and those of finite type to another paper, in which we plan to describe examples of packings carrying no cusp-uniform group action.

Finally we may consider a way of constructing an invariant system of cut sets in a tight packing: given a packing $X = P(\Gamma)$ of finite type, one immediately observes a map of $I(\Gamma)$ into the family $\mathcal{F}$ of all cut sets arising as intersections of $X$ with the circles $C'_R$ described above. Note that the collection of all such circles is not a packing, and that separation relations among these circles are well encoded in a tree (due to the given imbedding of $X$ in the 2-sphere).
The rest of work is based on a method for reconstructing this tree from the topology of $X$ as exemplified by the family $\mathcal{F}$, without any allusion to the particular embedding of $X$ in the 2-sphere. The difficulty in providing such a method lies – contrary to the case of cut-pairs for word-hyperbolic groups – mainly in that $\mathcal{F}$ is a non locally-finite family of intersecting cut sets: as we shall see, this difference is mostly due to the abundance in $G(\Gamma)$ of parabolic isometries; the fixed point of any such isometry attracts all finite cut sets. Our notion of the natural separation order on finite minimal cuts in $X$, introduced in section 3 and shown to satisfy the axioms of a pretree (below), is a way of overcoming this difficulty.

### 2.3 Pretrees

We have already mentioned that our technical approach to the problem of constructing a splitting tree for a group $G$ acting on a locally connected continuum $X$ is based on the idea of defining a betweenness relation on some family of cuts.

The appropriate notion of betweenness is that of a pretree due to Bowditch (see for instance [2]). In this section we review the definitions and show how to construct simplicial trees from special kinds of pretrees.

A set $T$ endowed with a ternary relation $a * b * c$ is said to be a pretree, if the following properties hold for all $a, b, c, d \in T$:

- (T1) $a * b * c \Rightarrow a \neq c$;
- (T2) $a * b * c \Rightarrow c * b * a$;
- (T3) $a * b * c \Rightarrow \neg b * a * c$;
- (T4) $a * b * c, d \neq b \Rightarrow a * b * d \lor d * b * c$.

It is customary to interpret the relation $a * b * c$ as “$b$ lies between $a$ and $c$”, this making axioms (T3) and (T4) intuitively responsible for the “treeness” of $T$. The first example of a pretree structure is the cut point order on a connected Hausdorff space: If $X$ is such a space, then write $a * b * c$ whenever there exists a clopen subset of $X \setminus \{b\}$ containing the point $a$ but not the point $c$. If $X$ is locally connected, then this is equivalent to $a$ and $c$ lying in distinct components of $X \setminus \{b\}$. This pretree structure on $X$ will be referred to as “the cut point structure” of $X$. This order structure allows us to view $X$ as a system of “fat chunks”, glued together using a treelike skeleton of points.

Let now $a, b$ be distinct points of a pretree $T$. Then the (oriented) interval $(a, b)_T$ is the set of all $x \in T$ satisfying $a * x * b$, with the linear ordering
defined by:

\[ x < y \iff a \ast x \ast y \ ; x, y \in (a, b)_T. \]

Closed intervals are defined by \([a, b]_T = \{a, b\} \cup (a, b)_T\), with the obvious extension of the ordering. If all intervals of a pretree \(T\) are finite, then \(T\) is said to be discrete.

Given \(a, b, c \in T\), the median \(\text{med}_T(a, b, c)\), when it exists, is defined to be the (unique) point \(z \in T\) lying in the intersection \([a, b]_T \cap [a, c]_T \cap [b, c]_T\). The subscript \(T\) will henceforth be omitted whenever the pretree is understood.

The simplicial analog of a median is, naturally, a vertex. When medians are not guaranteed, a useful notion is the notion of a star: a subset \(\sigma\) in a pretree \(T\) is null, if \(a \ast t \ast b\) holds for no \(a, b \in \sigma, t \in T\); such a set \(\sigma\) is said to be full, if given any \(t \in T \setminus \sigma\) there are \(a, b \in \sigma\) such that \(a \ast b \ast t\) holds; a maximal null subset in \(T\) is called a star, and a star having exactly two elements will be called a gap. The above notions should be attributed to Bowditch (compare with [2]), though here the definitions are slightly adopted to suit our special needs.

**Remark:** suppose \(\sigma\) is a null subset, and \(t \in T \setminus \sigma\). Then, if \(a \ast b \ast t\) holds for some \(a, b \in \sigma\), take any \(a' \in \sigma \setminus \{b\}\), and observe that then we have \(a' \ast b \ast t\) or \(a \ast b \ast a'\), where the latter relation is actually impossible, because \(\sigma\) is null. In particular, if \(\sigma\) has at least three points, then –

\[
\forall_{a, a' \in \sigma} \ a \ast b \ast t \ \Rightarrow \ b = \text{med}(a, a', t),
\]

and so \(b\) is uniquely-determined by \(t\) and \(\sigma\). This element \(b\) will be called the projection \(\pi_\sigma(t)\) of \(t\) to \(\sigma\).

The uniqueness of a projection implies that two distinct stars intersect in at-most one point: indeed, if \(\sigma, \tau\) are distinct stars, fix \(s \in \sigma \setminus \tau\), \(t \in \tau \setminus \sigma\) and assume \(a \in \sigma \cap \tau\), then, by the remark above, it will be enough to show \(s \ast a \ast t\), which is straightforward from the definitions.

If the given pretree \(T\) is discrete, then one is able to embed it in a simplicial tree by considering the union of \(T\) with the set of all stars of \(T\) as a set of vertices (see [2]). In the current work we will only have to deal with a special case, so that an even more explicit construction may be carried out:

**Definition 2.7 (bivalent pretree)** A pretree \(T\) (not necessarily discrete) is bivalent, if every \(t \in T\) is contained in precisely two distinct stars.

The forest \(\overline{T}\) associated to \(T\) is the graph having the stars of \(T\) for its vertices, with two stars being joined by an edge iff they intersect. Thus, we may identify \(T\) with the edge set of \(\overline{T}\).
One of course needs to justify the terminology for $\overline{T}$. For this we need to show that no reduced edge-path $\alpha$ in $\overline{T}$ is a cycle. First, one proves this for reduced edge-paths of length 3 by observing that any reduced edge path of the form $(e_1, e_2, e_3)$ satisfies the relation $e_1 \ast e_2 \ast e_3$. Next, an easy induction argument on the length of $\alpha$ employing the above observation finishes the proof. Finally, for the discrete case this amounts to –

**Proposition 2.8 (discrete bivalent pretree)** Suppose $T$ is a discrete bivalent pretree. Then $\overline{T}$ is a simplicial tree and the inclusion map $T \hookrightarrow \overline{T}$ is a pretree embedding.

In the future it will be more convenient to replace $\overline{T}$ by its first barycentric subdivision $\overline{T}^{(1)}$. This enables us to naturally direct the edges of the tree: every directed edge corresponds to an ordered pair of the form $(t, \sigma)$, where $t \in \sigma$ and $\sigma$ is a star in $T$. From now on, by $\overline{T}$ we shall mean this subdivided and directed version of the tree of $T$.

### 3 Separation in locally connected continua

Recall that a topological space $X$ is locally connected if and only if every component of every open subspace is open in $X$. If $X$ is a connected, locally connected and locally compact space, a closed subset $A \subset X$ is a cut in $X$, if $X \setminus A$ is disconnected; the cut $A$ is a minimal cut, if no proper closed subset of $A$ is a cut; the set of complementary (open) components of a cut $A$ is denoted by $C(A)$, and $N(A)$ denotes the cardinality of $C(A)$.

In this section, we shall develop a language to deal with finite minimal cuts in a henceforth fixed locally connected metrizable continuum $X$ having the property that no singleton in $X$ is a cut ("no global cut point condition").

A point $p \in X$ is a local cut point, if the space $X \setminus \{p\}$ has more than one end. The valence $\operatorname{val}(p)$ of $p$ is the number of ends of the punctured space $X \setminus \{p\}$. For example, if $A$ is a finite minimal cut (even a singleton), then all points of $A$ are local cut points, and it is also easy to see that $N(A) \leq \operatorname{val}(p)$ holds for all $p \in A$. The notion of a local cut point turned out to be of major importance in the splitting theory of word-hyperbolic groups when formulated in terms of their boundaries ([2]).

Following [2], let $X(2)$ henceforth denote the set of all $p \in X$ with $\operatorname{val}(p) = 2$ – the bivalent points in $X$ –, and let $X(3+)$ be the set of all point of valence at least 3 – these are the multivalent points of $X$. 

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3.1 The finiteness lemma

Our main tool for proving discreteness results about families of finite minimal cuts is, actually, a generalization of the notion of an annulus as presented in [2], but designed to make a more subtle use of the local connectedness of the space in special situations not arising in the splitting theory of uniform group actions.

Our main concern is to understand the way a system of cuts is embedded in the ambient space $X$. The examples at the end of this paragraph will justify the need for the following elementary definitions and results.

**Definition 3.1** Let $Y$ be a connected, locally connected and locally compact space. A compactification $\tilde{Y}$ of $Y$ is good, if $\tilde{Y}$ is locally connected, and $Y$ is open in $\tilde{Y}$.

**Proposition 3.2 (finiteness lemma)** Let $Y$ be a connected, locally connected and locally compact space, and let $A = A_1 \cup A_2 \subset Y$ be the union of two disjoint closed sets such that every $U \in C(A)$ has compact closure in $Y$. Further, let $\tilde{Y}$ be a good compactification of $Y$ such that $A_1$ and $A_2$ are closure-disjoint in $\tilde{Y}$. Then, the set $C(A_1; A_2)$ of all components $U \in C(A)$ with boundaries intersecting both $A_1$ and $A_2$ is a finite set.

Let us examine the given situation stripped of its combinatorial aspects:

**Lemma 3.3** Suppose $Y$ is a connected, locally connected and locally compact space, and $A$ is a closed set in $Y$ such that every $U \in C(A)$ has compact closure in $Y$. Then, the boundary $\tilde{Y} \setminus Y$ of every good compactification $\tilde{Y}$ of $Y$ lies in the closure of $A$ in $\tilde{Y}$.

The proof is a standard argument in elementary point-set topology, so let us focus instead on how the proposition uses this lemma:

**Proof of proposition 3.2:** For any set $B$ in $Y$, let $\bar{B}$ denote the closure of $B$ in $\tilde{Y}$. We can find disjoint open sets $U_{1,2}$ in $\tilde{Y}$ containing $\bar{A}_1$ and $\bar{A}_2$, respectively. Now, the family $\mathcal{U} = \{U_1, U_2\} \cup \mathcal{C}(A)$ has the property that any proper subfamily of $\mathcal{U}$ covering $\tilde{Y}$ contains $\mathcal{C}(A_1; A_2)$: note that no point of $A$ is covered by a $U \in \mathcal{C}(A)$, and no element $U \in \mathcal{C}(A_1; A_2)$ lies in $U_1 \cup U_2$, for otherwise the connectedness of $U$ implies $U$ is contained in $U_1$, say, forcing $\partial U$ to be disjoint from $A_2$ – a contradiction. Thus, any $\mathcal{V} \subseteq \mathcal{U}$ contains $\mathcal{C}(A_1; A_2)$.

To close, the lemma above implies $\mathcal{U}$ is, indeed, an open covering of $\tilde{Y}$; since $\tilde{Y}$ is compact, this shows $\mathcal{C}(A_1; A_2)$ is finite. ■
Applying the finiteness lemma to our special continuum \( X \) by substituting \( X \) for \( Y \) (and hence \( \tilde{Y} \) also equals \( X \)), we derive the following

**Corollary 3.4** If \( A \subset X \) is a finite minimal cut, then \( \mathcal{C}(A) \) is finite, and \( \partial U = A \) for all \( U \in \mathcal{C}(A) \). ■

The last corollary employs the finiteness lemma trivially, as \( \tilde{Y} \) contains no new information (being \textit{a priori} equal to \( Y \)). In order to understand what this lemma really has to say about the way finite minimal cuts are organized in a space \( X \), we consider two toy examples, both reminiscent of the “comb spaces” of basic topology courses:

**Example 3.5 (A simple alternative)** For each \( k \in \mathbb{N} \), let \( a_0^k = (2^{-k}, 0) \) and \( a_1^k = (2^{-k}, 1) \), and then for each \( k \) choose points \( p_i^k \in (0, 1] \times \{i\} \) \((i = 0, 1)\), each lying strictly in between \( a_i^k \) and \( a_{i+1}^k \). Consider the space

\[
Y_1 = ((0, 1] \times \{0, 1\}) \cup \bigcup_{k \in \mathbb{N}} [a_0^k, a_1^k]
\]

Here is a somewhat cumbersome way of proving that the closure \( \tilde{Y}_1 \) of \( Y_1 \) in the plane is not locally connected: the sets \( A_1 = \{p_0^k\} \) and \( A_2 = \{p_1^k\} \) satisfy the conditions of the finiteness lemma with respect to \( \tilde{Y}_1 \), but in this case we have \( \mathcal{C}(A) = \mathcal{C}(A_1; A_2) \), and it is not finite. Since \( \tilde{Y}_1 \) is a compactification of \( Y_1 \), it follows that \( \tilde{Y}_1 \) is not locally connected (see fig. 1a).

![Figure 1](image)

Figure 1: a bound on the size of minimal cut sets in a “convergent sequence” yields an alternative: either the space is compact or it is locally connected

It seems that a crucial factor at work in this setting is the fact that the cut sets \( P_k = \{p_0^k, p_1^k\} \) had a bounded size. Were we to add more (but still finitely many) horizontal intervals of the form \([0, 1] \times \{y\} \) to \( Y_1 \), the behaviour of the previous example would not have changed. Letting the size of the sets \( P_k \) out of control, though, does change the situation dramatically (see fig. 1b):
Example 3.6 (unbounded combinatorics ⇒ no alternative) Let $Y_2$ be the space obtained from $Y_1$ by adding to it all intervals of the form $(0, 2^{-k}] \times \{n \cdot 2^{-k}\}$, where $n \in \{1, \ldots, 2^k - 1\}$. Once again, let $P_k$ be a sequence of minimal cuts arising as the intersection of $Y_2$ with the vertical line with equation $y = x_k$, where $2^{-(k+1)} < x_k < 2^{-k}$.

This time, the closure $\bar{Y}_2$ of $Y_2$ in the plane is locally connected and compact. Setting $A = \bigcup_{k \in \mathbb{N}} P_k$ as before, but there is no possibility of splitting $A$ in two in such a way that $A_1$ and $A_2$ are closure-disjoint in $\bar{Y}_2$ and are both infinite. In view of the lemma, this accounts for the fact that such splittings of $A$ with the property that $C(A_1; A_2)$ is infinite do, indeed, exist.

To summarize, the idea behind the finiteness lemma is, roughly, that in order for a continuum $X$ to admit a “combinatorially bounded infinite ladder” it is necessary that $X$ fails to be locally connected.

3.2 Nesting of finite minimal cuts

In this section we introduce a new ordering on finite minimal cuts, designed to capture “treelike properties” of the space $X$. The novelty in our construction is that it works well for intersecting cuts, whereas all previous constructions dealt only with families of pairwise disjoint cuts (compare with [17]). In terms of cusp-uniform group actions, this provides us with a new tool for analyzing the structure of parabolic subgroups.

Definition 3.7 (nesting) A pair $A, B$ of minimal cuts in $X$ is nested (denote by $A \parallel B$) if there exists $K \in \mathcal{C}(A)$ lying in $X \setminus B$. Otherwise we say that $A$ and $B$ are transverse (denoted $A \not\parallel B$). Extending this notion to a family $\mathcal{F}$ of minimal cuts, we say that $\mathcal{F}$ is nested (resp. transverse), if all its elements are pairwise nested (resp. transverse).

As $A \parallel B$ is equivalent to the existence of (a uniquely determined) component $C_{AB} \in \mathcal{C}(B)$ containing $A$ in its closure, any component $K \in \mathcal{C}(B)$ other than $C_{AB}$ is disjoint from $A$, and so is contained in a component $K \setminus C(A)$; since $B$ is the boundary of $K$, it is closure-contained in $\bar{K}$, which, therefore, rightfully deserves the name $C_{BA}$.

Using the symmetry of nesting, one easily shows that the following nesting relation holds whenever $A \parallel B$:

$$C_{AB} \neq K \in \mathcal{C}(B) \iff K \subset C_{BA}. $$

The nesting relation provides a simple tool for the analysis of nested families of cuts. Our main technical result is the following –
Theorem 3.8 (natural separation order) Suppose $\mathcal{F}$ is a nested family of minimal cuts. Then the ternary relation defined for $A, G, B \in \mathcal{F}$ by

$$A \ast G \ast B \iff C_{AG} \subset C_{GB}$$

is a pretree structure on $\mathcal{F}$ satisfying the property that $A \cap B$ contains $A \cap G$ whenever $A \ast G \ast B$ holds. ■

Proof: The axioms (T1) and (T2) are obvious. Let us prove (T3): assume that both $D \ast E \ast F$ and $E \ast D \ast F$ hold for some distinct $D, E, F \in \mathcal{F}$. Now, $D \ast E \ast F$ implies $C_{DE} \subset C_{EF}$; on the other hand, $E \ast D \ast F$ implies $F \ast D \ast E$, so $C_{FE} \subset C_{DE} \subset C_{EF}$, and we obtain a contradiction: $C_{EF}$ cannot contain points of $F$, while $C_{FD}$ contains at least one point of $F$. Note that we use the nesting assumption on $\mathcal{F}$ implicitly each time we write down a symbol of the form $C_{AB}$ for $A, B \in \mathcal{F}$.

To prove (T4), assume $D \ast E \ast F$ and take any $G \in \mathcal{F}$. We are to prove that at least one of $D \ast E \ast G$ or $G \ast E \ast F$ holds, so assume $G \ast E \ast F$ does not, and we prove $D \ast E \ast G$: by the nesting relation,

$$\neg G \ast E \ast F \implies C_{GE} \nsubseteq C_{EF} \iff C_{GE} = C_{FE};$$

and therefore we have $G \ast E \ast D$, as desired. ■

Figure 2: The Apollonian packing in the plane (a) and on the sphere (b), with the point $p$ drawn at infinity. Note the relation $0 \ast 1 \ast 2$.

Example 3.9 (the Apollonian packing) Let $\Gamma'_1$ be the 1-skeleton of the regular tetrahedron inscribed in the unit 2-sphere, and let $\Gamma_1$ be its image
on the 2-sphere under the projection from the origin. Then the tight circle packing of type \( \Gamma_1 - X = \mathcal{P}(\Gamma_1) \) is the Apollonian packing, and the related group \( G_1 = G(\Gamma_1) \) is known to be a \( \mathbb{Z}_2^4 \), with the inversions \( I(\Gamma) \) being the corresponding generators. Note the 1 : 1 correspondence between the inversions in \( G(\Gamma) \) and the family \( \Theta \) of all cut-triples stabilized by these inversions. The treelike structure of the family \( \Theta \) is seen better when we use the inversion circles to illustrate the topological definition of betweenness (see fig. 2).

![Figure 3: Two packings and the graphs generating them (with one vertex at infinity).](image)

**Example 3.10 (general finite type packing)** Taking another type of packing – here \( X = \mathcal{P}(\Gamma) \), as shown in fig.3a – and setting \( \Theta \) again to be the family of all cut-triples in \( X \), we observe that, contrary to the previous example, not all of the reflections \( I(\Gamma) \) are accounted for in \( \Theta \), so it seems natural to augment the arising “treelike structure” by, say, adding to \( \Theta \) the \( G(\Gamma) \)-orbit of the 4-cut depicted in fig.3a. This, however, is not a good solution, as the resulting family is not a nested family.

Figure 3b depicts another example of a finite type packing in which cuts of order greater than 3 play a significant role, but here not all of them correspond to reflections in the corresponding reflection group (the marked circle always exists, but is never orthogonal to the packing).
3.3 A discreteness result for nested families

We are now ready for the analysis of general nested families of minimal cuts with “bounded combinatorics” like we have seen in example 3.5 (as opposed to example 3.6). Let us make things precise.

**Definition 3.11 (arc system between cuts)** Suppose $A, B$ are distinct nested finite minimal cuts in $X$. An arc system of size $k$ from $A$ to $B$ is a collection $\Gamma = \{\gamma_i\}_{i=1}^k$ of interiorly disjoint arcs, each joining a point of $A$ to a point of $B$, with the (relative) interior of each $\gamma_i$ lying in the space $C_{AB} \cap C_{BA}$. We denote the subset of point of $A$ (resp. $B$) lying on an element of $\Gamma$ by $\Gamma \cap A$ (resp. $\Gamma \cap B$); if $A = \Gamma \cap A$, we say $\Gamma$ covers $A$.

**Note:** each element of an arc system from $A$ to $B$ intersects each of $A$ and $B$ precisely once.

**Lemma 3.12** Suppose $A, B$ are distinct nested finite minimal cuts in $X$, and suppose every cut separating $A$ from $B$ in $X$ is of size at-least $k$. Then there exists an arc-system $\Gamma$ of size $k$ from $A$ to $B$. Moreover, if $A$ (or $B$) is of size $k$, then $\Gamma$ can be chosen so as to cover $A$ (resp. $B$).

This lemma is an immediate consequence of the cyclic connectivity theorem in one of its stronger forms:

**Theorem 3.13** ([13]) Suppose $Y$ is a locally connected metrizable continuum, let $a, b \in Y$ be distinct points and let $k \in \mathbb{N}$. If $Y$ contains no $(k-1)$-cut separating $a$ from $b$ in $Y$, then there exist $k$ interiorly-disjoint arcs from $a$ to $b$ in $Y$.

**Proof of Lemma 3.12:** Consider the space $Y$ obtained from $C_{AB} \cap C_{BA}$ by identifying all the points of $A$ to a single point $a$ and all points of $B$ to a point $b$. $Y$ a Peano continuum, and our hypotheses allow us to apply the cyclic connectivity theorem in the stated form to produce a system $\bar{\Gamma}$ of $k$ interiorly-disjoint arcs from $a$ to $b$ in $Y$. We then lift the arcs of $\bar{\Gamma}$ to produce the desired arc system in $X$.

For the second part of the lemma, assume $|A| = k$, and apply another construction for the space $Y$, the rest of the proof being left unchanged: instead of identifying the points $\{a_1, \ldots, a_k\}$ of $A$ to one point, glue a $k$-pod $K_{k,1}$ (a complete bipartite $(k,1)$ graph) to $C_{AB} \cap C_{BA}$ identifying each point $a_i$ with a unique leaf of $K_{k,1}$; do the same for $B$, if it has the appropriate size. ■

Let us now formulate the required discreteness result:
Theorem 3.14 (discreteness condition for $k$-cuts) Let $X$ be a Peano continuum such that every arc in $X$ is a nowhere-dense subset. Suppose $A$ and $B$ are disjoint nested minimal cuts of sizes at least 2 and at most $k$ such that every cut separating $A$ from $B$ is of size at-least $k$, and suppose $F$ is a nested family of minimal $k$-cuts. Then, the set $(A, B) \subseteq F$ of all $F \in F$ satisfying $A * F * B$ is finite.

Remark: This theorem is readily applied to any $X$ arising as a limit set for a non-elementary convergence group $G$, unless $X$ is homeomorphic to the circle – in particular, if $X$ contains a minimal cut triple. Indeed, if an arc $\gamma$ contains an open subset $U$ of $X$, then $U$ may be taken to be homeomorphic to an open real interval. Find a point $q \in U$ which is the repelling fixed point of a loxodromic element $g \in G$, and denote the attracting fixed point by $p$. Substituting $g$ by a high enough power if necessary, we find out that $X$ is the one-point compactification of $Y = \bigcup_{n \in \mathbb{N}} g^n(U)$ by $p$, with $g^n(U) \subset g^{n+1}(U)$ for all natural $n$. Since $U$ is an open real interval, $Y$ is readily shown to be homeomorphic to $\mathbb{R}$, so that $X$ is homeomorphic to $S^1$.

Proof of Theorem 3.14: If $(A, B)$ is not empty, fix a $k$-cut $F \in (A, B)$, and we have to show both $(A, F)$ and $(F, B)$ are finite (recall that since $(A, B) \cup \{A, B\}$ is a nested family, every interval in the corresponding pretree has a natural linear structure).

Let us prove $(A, F)$ is finite. Let us fix an arc system $\Gamma$ of maximum possible size from $A$ to $F$; by lemma 3.12, we have $|\Gamma| \geq k$. Since every $\gamma \in \Gamma$ joins a point of $A$ to a point of $F$, it must pass through each $E \in (A, F)$; in particular, $|\Gamma| = k$ precisely, and $F$ is covered by $\Gamma$. Thus, writing $\Gamma = \{\gamma_i\}_{i=1}^k$ immediately enables one to write $E = \{e_i\}_{i=1}^k$ with $e_i \in \gamma_i$ for all $i \in \{1, \ldots, k\}$, as the $\gamma_i$ are interiorly disjoint. We keep this notation until the end of the proof. Consider two cases:

Case 1. $\Gamma$ covers $A$.

In this case, the idea is to apply the finiteness lemma directly: for each $i \in \{1, \ldots, k\}$, let $T_i$ be the closure of the set of all points $e_i$, $E$ ranging over all of $(A, B)$, and let $a(i)$ denote the point of $A$ lying on $\gamma_i$. Since $\Gamma$ covers both $A$ and $F$, the sets $T_a = \bigcup_{a=a(i)} T_i$ are subsets of the space $Y = C_{AF} \cap C_{FA}$ which are closure-disjoint in the good compactification $\bar{Y}$ of $Y$ (note $T_i \subseteq \gamma_i$ and $\gamma_i$ are all closed disjoint arcs). Thus, by the finiteness lemma (3.2) – which can be applied as $|A| \geq 2$ –, the number of components in $Z := Y \setminus \bigcup_{i=1}^k T_i$ is finite, and we denote it by $N$. 19
Suppose now \((A, F)\) contains \(N\) distinct elements \(E^1, \ldots, E^N\). Since \((A, F)\) is linearly ordered by the separation order, we may assume

\[ A < E^1 < E^2 < \ldots < E^N < F, \]

and we have \(e^j_i \in T_i\) for all \(i \in \{1, \ldots, k\}\) and \(j \in \{1, \ldots, N\}\). Thus, \(\tilde{Z} := Y \setminus \bigcup_{j=1}^{N} E_j\) contains \(Z\), meaning every component \(U\) of \(Z\) is contained in a unique component \(\tilde{U}\) of \(\tilde{Z}\).

We arrive at a contradiction by proving the map \(U \mapsto \tilde{U}\) is onto the set of components of \(\tilde{Z}\): indeed, if \(W\) is a component of \(\tilde{Z}\) not containing any component of \(Z\), then it constitutes an open subset of \(X\) covered by \(\bigcup_{i=1}^{k} T_i\) – contradicting the assumption that arcs in \(X\) are nowhere-dense.

**Case 2.** \(\Gamma\) does not cover \(A\).

Our aim now is to show that the cut system \((A, B)\) is bounded away from \(A\) in some sense; this will enable the application of the finiteness lemma in the same manner as in the previous case, finishing the proof.

Let \(T_i\) be defined as before, and let us split \(A\) as a disjoint union \(A = (A \cap \Gamma) \cup A'\). Fix an exhaustion \(\{K_n\}_{n=1}^{\infty}\) of \(C_{FA}\) by compact connected sets containing \(F\), take a system \(\{U_a\}_{a \in A}\) of pairwise-disjoint connected neighbourhoods of the respective points of \(A\) such that for \(a \in A'\) we have \(U_a \cap \gamma_i = \emptyset\) for all \(i \in \{1, \ldots, k\}\), and find a \(K_{n_0}\) containing \(C_{FA} \setminus \bigcup_{a \in A} U_a\).

Since \(K_{n_0}\) is disjoint from \(A\), we may find for each \(a \in A \cap \Gamma\) a neighbourhood \(U'_a \subset U_a \setminus K_{n_0}\), and consider the sets

\[ U = \bigcup_{a \in A \cap \Gamma} U'_a, \quad F = K_{n_0} \cup \bigcup_{a \in A'} U_a. \]

We note \(U\) and \(F\) are disjoint. Therefore, since \(F\) is a connected subset containing both points of \(B\) and points of \(A\), no \(E \in (A, F)\) is entirely contained in \(U\). Thus, there is an index \(i \in \{1, \ldots, k\}\) such that \(T_i\) is disjoint from some neighbourhood \(V \subset U\) of \(A\). Thus, the set \(T_i\) is closure-disjoint from the set \(\bigcup_{j \neq i} T_j\), and the finiteness lemma may again be applied to the spaces we had defined in case 1. ■

**Remark:** Note the usage of \(F\) being nested is not strong. There are two points to consider:

- First, one needs to know \((A, B)\) is finite if \((A, F)\) and \((F, B)\) are finite for at least some \(F \in \mathcal{F}\). For example, if the set of \(E \in (A, B)\) satisfying \(E \parallel F\) is finite, then we may proceed with the above proof.
Second, one needs to be able to argue that if \((A, F)\) is infinite, then there exist finite subfamilies \(\{E_j\}_{j=1}^m\) of \((A, F)\) with \(C(E_1 \cup \ldots \cup E_m)\) arbitrarily large in \(Y\).

In view of the intended applications, only the first point seems really significant. Thus, this discreteness result may be applied to some non-nested families as well.

### 3.4 Exactness

A special type of cut pairs has played a central role in the splitting theory of uniform actions. As we shall immediately observe, the cut triples in the last examples enjoy the same property:

**Definition 3.15 (exact cuts)** A finite minimal cut \(A\) in \(X\) is said to be exact, if \(N(A) = \text{val}(p)\) for all \(p \in A\).

Note that, by definition, an exact cut has to be finite and minimal. In particular, if \(A\) is an exact cut, then each \(K \in \mathcal{C}(A)\) has precisely \(|A|\) ends. Equivalently, for each \(p \in A\) there is a bijective correspondence between the ends of \(X \setminus \{p\}\) and \(\mathcal{C}(A)\) (if \(A\) is not exact, several distinct ends of \(X \setminus \{p\}\) may be the ends of the same \(K \in \mathcal{C}(A)\)).

The particular family of cuts studied by Bowditch in [2] was the family of all exact cut pairs, when the motivating example is that of a pair of the form \(\text{Fix}(g)\) for a loxodromic element \(g \in G\) where \(G\) is a convergence group acting minimally on \(X\): by now it is common knowledge that \(\text{Fix}(g)\) is an exact cut whenever it cuts \(X\). It follows from the next lemma that any pair \(A, B\) of exact cut pairs is nested unless both \(A\) and \(B\) are bivalent.

**Lemma 3.16 (transversal separation lemma)** Suppose \(A\) and \(B\) are transverse finite minimal cuts. Then, for all \(u \in \mathcal{C}(B)\), \(A \cap u\) is a cut in \(u\) with at-least \(N(A)\) complementary components. Moreover, if, in addition, \(B\) is exact, then \(A \cap u\) separates \(B = \partial u\) in \(\overline{u}\).

**Proof:** Let \(u \in \mathcal{C}(B)\), and let \(v \in \mathcal{C}(A)\) if \(u \cap v = \emptyset\), then \(A = \partial v\) is disjoint from \(u\) – hence \(u\) lies in a component of \(X \setminus A\), contradicting transversality. Thus, we have shown that every component \(v \in \mathcal{C}(A)\) intersects \(u\), so that \(u \setminus A\) (which is open) is the union of the non-empty open sets \(u \cap v\), where \(v\) ranges over all elements of \(\mathcal{C}(A)\), and we have proved the first assertion.

Suppose now that \(B\) is exact, but assume the second assertion of the lemma is wrong. Since \(\overline{u}\) is a locally connected continuum, there exists a closed
connected set $F$ of $\mathfrak{u}$ containing $B = \partial u$ and contained in $\mathfrak{u} \setminus (A \cap u)$. Since $C(A \cup B)$ is finite, there exists a $v \in C(A)$ containing $F$ in its closure. The exactness of $B$ (which implies that every point of $B$ is univalent in $\mathfrak{u}$) then tells us that $v$ is the only component of $X \setminus A$ satisfying this condition. Thus, for any $v' \in C(A)$ other than $v$, any component of the set $v' \cap u$ has its boundary (in $X$!) contained in $A \cap u$. In particular, $A \cap u$ is a proper subcut of $A$ – contradiction. ■

In the case when $A, B$ are transverse (bivalent) exact cut pairs, it is easy to prove (see for instance [2]) that $A \cup B$ is contained in an equivalence class of the equivalence relation defined on $X(2)$ by:

$$x \sim_2 y \iff x = y \text{ or } \{x, y\} \text{ is a cut in } X$$

We denote by $\Sigma$ the set of all non-singleton ($\sim_2$)-classes. A point of $X(2)$ whose ($\sim_2$)-class is a singleton will be called a rigid point. In addition, we fix the following notation for the main players in this study:

- $\Theta$ – the set of all exact multivalent cut pairs;
- $\Xi_3$ – the set of all bivalent exact cut triples.

There are at least three reasons for considering $\Xi_3$ in the context of cusp-uniform group action on locally connected continua: first, the examples we have seen (circle packings) point to a possible connection between splitting properties of the group and the structure of bivalent exact cut triples in the space it acts upon; second, the same examples show that the allowed presence of parabolic transformations produces local cut point structures which are impossible in the setting of a uniform action, as will be demonstrated in the next two sections; third, an element of $\Xi_3$, being a point of $3X$, has a finite stabilizer in any convergence group of $X$, this making it a natural choice for the role of a “reason for splitting over a finite subgroup”.

4 Effects of a cusp-uniform action

From now on let us suppose the space $X$ admits a minimal cusp-uniform convergence group action by a group $G$. Let us recall that the difference between this setting and the uniform one is in our allowing $G$ to have bounded parabolic points in $X$ (the rest being conical limit points). In view of this, the following facts have been proven in [2] for uniform group actions, and, using theorem 2.6, one easily checks that the same proofs apply for the cusp-uniform setting verbatim:
Proposition 4.1 A global cut point of $X$ is a parabolic point.

Proposition 4.2 The valence function is uniformly bounded on the set of conical limit points.

Let us now briefly explain how the proofs of [2] are brought to apply to the cusp-uniform setting.

4.1 Conical points

Upon close examination of [2], one finds that the essential ingredient for analyzing the properties of local cut points of a space $X$ admitting a uniform action by a group $G$ is an annuli system. Such a system is constructed there in lemmas 5.1 and 5.2.

In the cusp-uniform situation one replaces those with the following. Given a compact metrizable perfect space $X$ and a cusp-uniform action by a group $G$, fix a compact subset $K$ of $3X$ as guaranteed by theorem 2.6. By compactness and local connectivity, there exists a finite covering of $K$ by open sets of the form $U_i \times V_i \times W_i$ ($i \in \{1, \ldots, p\}$) satisfying the requirement that for each $i$ the sets $U_i, V_i, W_i$ are all connected and pairwise closure-disjoint.

Then, the following variation of lemma 5.2 of [2] holds, with the same proof as in [2] (though based on the special property of $K$):

Lemma 4.3 (conical annuli system) There exist finite collections $(U_i)_{i=1}^p$ and $(O_i)_{i=1}^p$ of open connected subsets of $X$ such that $U_i \cap \overline{O_i} = \emptyset$ for all $i$, and such that if $F \subset M$ is closed and $x \in X \setminus F$ is a conical limit point of $G$, then there exist $g \in G$ and $i \in \{1, \ldots, p\}$ such that $g(x) \in U_i$ and $g(F) \subseteq O_i$.

In order to obtain proofs of our propositions 4.1, 4.2, 4.7 and 4.8, one needs only to proceed (using the systems $(U_i)$ and $(O_i)$ of the lemma above) using the very arguments used in [2].

4.2 Parabolic points

Since the above results (as well as propositions 4.7 and 4.8) do not extend to the set of parabolic points, the need arose for us to prove the following “ping-pong” type result:
Proposition 4.4 Suppose $p \in X$ is a parabolic non-cut point of $X$, and let $X(p)$ be the end compactification of $X \setminus \{p\}$. Then the action of the parabolic group $G_p$ on $X \setminus \{p\}$ extends to a convergence group action on $X(p)$, and if $p$ is bounded, then $\mathcal{E}(X \setminus \{p\})$ equals the limit set of this action. In particular, if, in addition, $p$ is of finite valence, then $\text{val}(p)$ is at most 2.

In order to gain some understanding of what goes on here – the role played by the local connectivity assumption being especially important – let us prove separately the weaker result that if $\text{val}(p)$ is finite, then it is less than or equal to 2, and delay the proof of the general statement.

Proof of the valence bound: Let $K$ be a compact connected set separating all the points of $\mathcal{E} = \mathcal{E}(X \setminus \{p\})$ from each other. Such a set exists, because $X \setminus \{p\}$ is a connected, locally compact and locally connected space.

For each end $\epsilon \in \mathcal{E}$ let $C_\epsilon$ denote the component of $\epsilon$ in $X(p) \setminus K$. Now, since $\mathcal{E}$ is finite, there exists a finite-index subgroup $H$ of $G_p$ fixing $\mathcal{E}$ pointwise. Being an infinite subgroup of $G_p$, $H$ acts properly-discontinuously on $X \setminus \{p\}$, so there exists $h \in H$ with $h(K)$ disjoint from $K$. Thus, there is an end $\epsilon \in \mathcal{E}$ such that $h(K)$ is contained in $C_\epsilon$.

Finally, suppose $\alpha, \beta \in \mathcal{E} \setminus \{\epsilon\}$ are distinct ends. Then the set $C_\alpha \cup K \cup C_\beta$ is a connected subset of $X(p) \setminus h(K)$. On the other hand, $h$ is a homeomorphism of $X(p)$ onto itself keeping $\mathcal{E}$ pointwise fixed, and since $K$ separates $\alpha$ from $\beta$, so does $h(K)$ – a contradiction. ■

Another corollary of the last proposition is a partial answer to a question of Tukia [16], regarding sufficient conditions for the existence of parabolic elements in the stabilizer of a parabolic point:

Corollary 4.5 (Existence of parabolics) Let $p$ be a bounded parabolic point of a convergence group $G$ of a Peano space $X$. If $p$ is a non-cut point of $X$ of valence at-least 2, then $p$ is a parabolic fixed point.

Proof of the corollary: In both possible cases, the group $\hat{G}_p$ contains a loxodromic $\hat{g}$, and so the corresponding element $g \in G_p$ is a parabolic element - as desired. ■

The above propositions, as well as known results about word-hyperbolic groups (see e.g. [15]) indicate that in order for our setting to be amenable to “discrete” analysis (using combinatorial structures such as trees), we need to work under the additional assumption that $X$ has no global cut point and no point of infinite valence.
4.2.1 Proof of proposition 4.4

Let us focus on the properly discontinuous action of \( G_p \) on the space \( X^* = X \setminus \{p\} \).

We recall a way to construct \( \mathcal{E}X^* \) in our setting. Given any compact \( K \subset X^* \), denote by \( C_\infty(K) \) the set of all components of \( X^* \setminus K \) having non-compact closure in \( X^* \); one shows \( C_\infty(K) \) is always finite. Fixing any exhaustion \( \{K_n\}_{n=1}^\infty \) of \( X^* \) by compact sets, we define an end of \( X^* \) to be a descending sequence \( \epsilon = (C_n)_{n=1}^\infty \) satisfying \( C_n \in C_\infty(K_n) \) for all \( n \); for all \( n \), we say \( \epsilon \) is covered by \( C_n \). Finally, we topologize the space of all ends as follows: for each \( n \) and each \( C_n \in C_\infty(K_n) \) we denote by \( \tilde{C}_n \) the union of \( C_n \) with the set of all ends it covers.

One shows that the space \( \mathcal{E} := \mathcal{E}X^* \) and the topology just defined on \( X(p) = X^* \cup \mathcal{E} \) do not depend on the particular choice of the exhaustion \( K_n \). In particular, the assumption of local connectivity allows us to assume all the \( K_n \) are connected, and a simple calculation shows one may assume \( C(K_n) = C_\infty(K_n) \) for all \( n \).

Since \( G_p \) acts properly discontinuously on \( X^* \), the following situation should be quite frequent:

**Lemma 4.6** Let \( K \) be a connected compactum in \( X^* \) such that all elements of \( C(K) = C_\infty(K) \), let \( C \in C_\infty(K) \) and \( g \in G_p \) such that \( gK \subset C \). Then there exists a unique component \( C^{\text{op}}(g) \in C_\infty(K) \) such that for all \( C' \in C_\infty(K) \) one has

\[
C' \neq C^{\text{op}}(g) \iff gC' \subseteq C.
\]

Furthermore, \( C^{\text{op}}(g) \) is characterized by its containing the set \( g^{-1}(K) \).

**Proof:** We consider \( K \) as a subspace of \( X^* - gK \); since \( K \) is connected, there exists a unique element \( D \in C(gK) \) containing \( K \). Now, \( D \) is unbounded, and we set \( C^{\text{op}}(g) = g^{-1}D \). Checking this choice is the correct one is straightforward. ■

Let us now prove proposition 4.4. Let \( \hat{G}_p \) be the group of homeomorphisms of \( X(p) \) arising as the unique extension of the action of \( G_p \) on \( X^* \) to the endspace \( \mathcal{E} \). Since \( G_p \) is a convergence group of the space \( X \), it is enough to show that a sequence \( (g_n)_{n=1}^\infty \) in \( G_p \) converging to \( p \) uniformly on compacts in \( X^* \) contains a subsequence with indices \( (n_k)_{k=1}^\infty \) such that \( (\hat{g}_{n_k})_{k=1}^\infty \) converges to an end \( \alpha \in \mathcal{E} \) uniformly on compacts in the space \( X(p) \setminus \{\beta\} \), where \( \beta \) is some (possibly other) element of \( \mathcal{E} \).
We start off with the sequence \((g_n)_{n=1}^\infty\), and fix an exhaustion \(\{K_n\}_{n=1}^\infty\) of \(X^*\) by connected compacta satisfying \(\mathcal{C}(K_n) = \mathcal{C}_\infty(K_n)\) for all \(n\). Passing to a subsequence, we may assume that \(g_n(K_n) \cap K_n = \emptyset\) for all \(n \in \mathbb{N}\).

We now use an inductive process to single-out the required ends \(\alpha, \beta \in \mathcal{E}X^*\):

- Choose \(C_1 \in \mathcal{C}_\infty(K_1)\) such that the set \(A_1 = \{n \in \mathbb{N}| g_nK_n \subseteq C_1\}\) is infinite. \(C_1\) exists because \(n \geq 1 \Rightarrow K_1 \subseteq K_n \Rightarrow g_nK_1 \cap K_1 = \emptyset\), and thus there has to be a component in \(\mathcal{C}_\infty(K_1)\) containing \(g_nK_1\) for infinitely many \(n\), as \(\mathcal{C}_\infty(K_1)\) is finite.

Consider now the set of all \(\text{op}^1_1(g_n) \in \mathcal{C}_\infty(K_1)\): this is also a finite set, so there is a \(D_1 \in \mathcal{C}_\infty(K_1)\) such that \(D_1 = \text{op}^1_1(g_n)\) for infinitely many values of \(n \in A_1\). We denote the set of all such \(n\) by \(B_1\).

- Assume inductively that we have finite sequences \(\{C_i\}_{i=1}^m, \{D_i\}_{i=1}^m\) such that:

1. \(C_i, D_i \in \mathcal{C}_\infty(K_i)\) for all \(1 \leq i \leq m\)
2. the sets \(B_i = \{n \in \mathbb{N}| g_nK_i \subseteq C_i \land D_i = \text{op}^i_1(g_n)\}\) form a descending sequence of infinite subsets.

Then, there has to be \(C_{m+1} \in \mathcal{C}_\infty(K_{m+1})\) such that the set
\[
A_{m+1} = \{n \in B_m| g_nK_{m+1} \subseteq C_{m+1}\}
\]
is infinite, and for this \(C_{m+1}\) there has to be a
\[
D_{m+1} \in \mathcal{C}_\infty(K_{m+1})
\]
such that the set \(B_{m+1} = \{n \in A_{m+1}| D_{m+1} = \text{op}^m_{m+1}(g_n)\}\) is also infinite.

We set \(\alpha = (C_m)_{m=1}^\infty, \beta = (D_m)_{m=1}^\infty \in \mathcal{E}X^*\), and \(n_i = \text{min}B_i\), and contend that the sequence \((\hat{g}_{n_i})_{i=1}^\infty\) converges to \(\alpha\) uniformly on every compact subset of \(X(\hat{p}) - \{\beta\}\), thus proving the theorem. More specifically, a straightforward computation yields that, given a base neighbourhood \(\hat{C}_m\) of the end \(\alpha\), \(i \geq m\) implies
\[
\hat{g}_{n_i} \left( X(\hat{p}) - \hat{D}_m \right) \subseteq \hat{C}_m,
\]
which is what we claimed.

Finally, we have to address the additional claim regarding the case of a bounded parabolic point: to verify that every end \(\alpha \in \mathcal{E}X^*\) is the attracting
point of some convergence sequence \((g_n)_{n=1}^\infty\) in \(\hat{G}_p\), we first arrange for \(K_1\) to intersect every \(G_p\)-orbit in \(X^*\), and then, given an end \(\alpha = (C_n)_{n=1}^\infty\), we use the co-compactness of \(G_p\) on \(X^*\) to find elements \(g_n \in G_p\) such that \(g_nK_n \subseteq C_n\). We then apply the above construction to obtain a convergence subsequence \(\hat{g}_n\), which, obviously, has the end \(\alpha\) for its attracting point. ■

4.3 Characterizing exact cut pairs

Under the named additional assumptions, one is able to transfer more of the results of [2] into the cusp-uniform setting (see 4.1).

**Proposition 4.7 (multivalent implies loxodromic)** If a point \(p \in X\) has \(\text{val}(p) \geq 3\), then it is a loxodromic fixed point. In particular, \(p\) participates in a unique multivalent exact cut pair.

A variant for bivalent points of \(X\) exists as well:

**Proposition 4.8 (trichotomy of bivalent points)** Suppose \(X\) is a locally connected metrizable continuum not homeomorphic to \(S^1\) and carrying a cusp-uniform action by a group \(G\). If \(p\) is a bivalent point of \(X\), then precisely one of the following occurs:

- \(p\) is a parabolic point, and \([p]_2\) is a singleton;
- \(p\) is non-parabolic and \([p]_2\) is the fixed point set of a loxodromic element of \(G\);
- the closure of \([p]_2\) is a cyclically-ordered Cantor set in \(X\).

Note that the first two cases may be characterized by \(p\) being an isolated point of its 2-class.

Not only that the last two propositions are crucial for our constructions, but they also explain why no group may act uniformly on a circle packing: in the context of a uniform action, there are no parabolic points – hence every local cut point participates in a cut pair; since there are no cut pairs in a circle packing, a uniform action on it is impossible.

4.4 Exact cuts of higher order

The following lemma shows how two nested exact cuts may interact:
Lemma 4.9 If $A$ and $B$ are distinct nested exact cuts and one of them is multivalent, then they are disjoint.

**Proof:** Let $p \in A \cap B$; it is enough to show that $\text{val}(p) = 2$. By the exactness of $B$, every element of $C(B)$ has exactly one end at $p$. In particular, every $K \in C(B), K \neq C_{AB}$ has exactly one end at $p$, so the nesting condition implies that all the ends of $X$ at $p$ but one are ends of $C_{BA}$. But the exactness of $A$ tells us that $C_{BA}$ too has only one end at the point $p$, so $X$ is two-ended at $p$. ■

In particular, we immediately deduce –

**Corollary 4.10** Suppose $A, B \in \Theta$ are multivalent exact cut pairs. Then either $A = B$ or $A \cap B = \emptyset$. ■

– a lemma proved previously in [2] by other means.

The additional assumption about the presence of a cusp-uniform group action on $X$ turns out to influence the structure of exact cuts of high order (i.e. containing at least three points), giving them a role similar to that of Jordan curves on the 2-sphere:

**Proposition 4.11** If $A$ is an exact cut of order greater than 2, then $A$ is bivalent.

**Proof:** Assume $A$ has valence $v \geq 3$. Fix any $p \in A$: since $\text{val}(p) > 2$, proposition 4.7 provides us with an exact cut pair $P$ containing $p$.

Now, since $A$ and $P$ are distinct and nested, lemma 4.9 implies $N(A) = N(B) = 2$ – a contradiction. ■

This, together with the transversal separation lemma, gives us a clear picture of how exact 3-cuts in $X$ are arranged:

**Proposition 4.12** If $A, B \in \Xi_3$ are transverse, then both $A$ and $B$ intersect the same non-singleton ($\sim_2$)-class.

**Proof:** Apply the transversal separation lemma (lemma 3.16) to the pair $A, B$. Since $B$ is bivalent and $A$ is a transverse exact 3-cut, there is a component $u \in C(B)$ such that $A$ intersects $u$ in a unique point – $p$, say – cutting $\overline{u}$ between the points of $B$. Since $|B| = 3$ and $\text{val}(p) = 2$, $u \setminus \{p\}$ has precisely one bivalent component (see figure 4). ■ 

In the absence of bivalent exact cut pairs in $X$ this means that $\Xi_3$ is a pretree, by theorem 3.8. Actually, the objective of this paper is to prove in
this case that $\Xi_3$ is a discrete bivalent pretree, giving rise to a topologically-defined simplicial tree $\Xi_3$. We will show that any group $G$ acting in $X$ as a cusp-uniform convergence group acts cofinitely on this tree with finite edge-stabilizers, where the midpoints of edges correspond to the elements of $\Xi_3$. Thus, if $\Xi_3$ is non-empty, then $G$ splits over a finite subgroup.

On the other end of the scale, if $X$ does contain non-singleton ($\sim_2$) classes, the natural separation order on $\Xi_3$ may well be non-discrete, as well as non-treelike, as the following diagram demonstrates: The situation depicted in

![Diagram](image)

Figure 4: a pair of transverse exact bivalent 3-cuts shares a non-trivial ($\sim_2$)-class.

![Diagram](image)

Figure 5: a pair of nested exact bivalent 3-cuts having infinitely-many intermediate 3-cuts.

fig.5 is, in some sense, a “degeneration of the 3-cut structure” into the 2-cut structure.

We shall briefly relate to these difficulties in section 6, since our relevant results are outside the scope of this paper. Let us close this section with a definition:

**Definition 4.13 (cut-rigid space)** Suppose $X$ is a locally connected metrizable continuum. We say that $X$ is a cut-rigid continuum, if $X$ contains no global cut point, no point of infinite valence and no exact bivalent cut pairs.

For example, any (topological) circle-packing is such a space.
5 The main theorem

We henceforth assume $X$ to be a cut-rigid space admitting a cusp-uniform group action by a group $G$, and such that $\Xi_3$ is non-empty. In particular, $X$ is not homeomorphic to $S^1$.

Let us precede the detailed formulation of our main theorem by the following observation. Suppose $p$ is a point belonging to some $A \in \Xi_3$. Since $A$ is a minimal 3-cut, we have $[p]_{\sim_2}$ is a singleton, so that prop.4.8 implies $p$ is a parabolic point. This calls for the definition:

**Definition 5.1 (idle parabolics)** Suppose $G$ is a cusp-uniform group acting on a cut-rigid space $X$. A parabolic point $p$ of $G$ is said to be idle, if $p$ is contained in no element of $\Xi_3$. The stabilizer of an idle parabolic point will be called an idle parabolic subgroup of $G$.

We are now in position to formulate the main theorem, of which Theorem A of the introduction is an obvious corollary:

**Theorem 5.2 (Theorem A)** Suppose $X$ is a cut-rigid continuum admitting a minimal cusp-uniform group action. Then:

1. if $\Xi_3$ is non-empty, then the pretree $\Xi_3$ is an infinite discrete bivalent pretree;

2. if a group $G$ acts on $X$ cusp-uniformly and minimally, then $G$ acts co-finitely and minimally on the (directed) tree $\Xi_3$, with finite edge stabilizers.

3. for each vertex $v$ of $\Xi_3$, there exists a parabolic transformation of $G$ outside $G_v$.

4. for each vertex $v$ of $\Xi_3$, the stabilizer $G_v$ of $v$ is infinite if and only if $v$ is an infinite star in $\Xi_3$.

5. every idle parabolic subgroup of $G$ stabilizes a vertex in $\Xi_3$.

Let us state some corollaries:

**Corollary 5.3** Suppose $G$ acts cusp-uniformly and minimally on a cut-rigid space $X$. If $X$ contains a bivalent cut triple, then $G$ has more than one end.

Geometrically, the following corollary is even more interesting:
Corollary 5.4 Suppose $G$ acts cusp-uniformly on a cut-rigid space $X$. If all stars of $\Xi_3$ in $X$ are finite, then $G$ is the fundamental group of a finite graph of finite groups. In particular, $G$ is a finitely generated virtually free group.

For example, if $X$ is a tight circle packing of finite type, then this corollary may be applied not only to the reflection group defined by the packing, but also to any cusp-uniform group of homeomorphisms of this packing. Thus, perhaps the geometric structure of the packing may actually be thought of as a topological property – see questions 3 and 4 in section 6 for a precise formulation of this idea.

5.1 Rigidity and the discreteness of $\Xi_3$.

We have already seen that under the rigidity assumption the combinatorics of systems of exact 3-cuts becomes treelike. In order to understand the structure of intervals in the pretree $\Xi_3$, we have to consider all possible interactions between two distinct elements of $\Xi_3$.

Suppose $A$ is a bivalent exact cut. Given any $K \in C(A)$ let us denote the second element of $C(A)$ by $K^*$.

Suppose now that $A$ and $B$ are nested exact $k$-cuts with $k \geq 3$. Then, using the nesting relation we obtain that $C_{AB}^*$ and $C_{BA}^*$ are disjoint. The minimality of $A$ and of $B$ implies every element $K \in C(A \cup B)$ other than $C_{BA}^*, C_{AB}^*$ has its closure intersect both $A$ and $B$. Thus, the intersection

$$C_{AB} \cap C_{BA} = X \setminus (C_{BA}^* \cup C_{AB}^*)$$

is the union of $C(A; B)$, and it will be denoted by $A \cdot B$.

Let us now define an equivalence relation on $A \cup B$: given $x, y \in A \cup B$ we deem them equivalent if both are contained in the closure of the same component of $A \cdot B$. The exactness assumption on $A$ and $B$ means that this is, indeed, an equivalence relation.

For $k \geq 4$, there is no problem for this relation to induce non-trivial partitions on both $A$ and $B$, but for $k = 3$ any nontrivial relation except one implies a contradiction to rigidity, and the only exception arises when $|A \cap B| = 1$ and $A \cdot B$ is connected. Thus, in any case $A \cdot B$ is connected (see fig.6).

We are now ready to prove the discreteness of $\Xi_3$ as a pretree (under the rigidity assumption). Suppose $A, B \in \Xi_3$ are distinct, and fix points $a \in C_{BA}^*$ and $b \in C_{AB}^*$. By the preceding discussion, there are two distinct cases to consider: the case when $A$ and $B$ are disjoint, and the case when their intersection is a point.
Let Proposition 5.5

Case 1 – $A \cap B = \{p\}$: Consider the space $X_{(p)}$ arising as the end compactification of $X \setminus \{p\}$. This space is a locally connected continuum, and we claim it has no cut point. If so, then we may immediately apply theorem 3.14 to the space $X_{(p)}$, the pair of cuts $A \setminus \{p\}, B \setminus \{p\}$ and to the nested family

$$\mathcal{F} = \{ E \setminus \{p\} \mid E \in (A, B)_{\Xi_3} \}$$

to conclude $(A, B)_{\Xi_3}$ is finite, as desired.

Suppose now $q$ is a cut point of $X_{(p)}$. This means $P = \{p, q\}$ is a minimal cut pair in the original space $X$. Since $val(p) = 2$, it is enough to show that $P$ is an exact cut pair in order to achieve a contradiction.

**Proposition 5.5** Let $X$ be a cut-rigid space admitting a cusp-uniform group $G$. Then every cut-pair $P$ in $X$ containing a bivalent point is exact.

**Proof:** Write $P = \{a, b\}$, and suppose $val(a) > 2$ and $val(b) = 2$. Since $val(a) > 2$, there is a loxodromic $g \in G$ such that $a = fix_+(g)$, and $fix(g)$ is an exact cut-pair. Passing to some power of $g$, we may assume that $g$ fixes the component $C \in \mathcal{C}(fix(g))$ containing $b$. In addition to this, if $U$ is a connected neighbourhood of $a = fix_+(g)$ in $X \setminus \{b\}$, then passing again to high powers of $g$ we may assume $g(b) \in U$.

Now, consider the continuum $\hat{C}$ with the cut point order: we have $a \ast b \ast fix_-(g)$ and so $a \ast g(b) \ast fix_-(g)$ (see fig.7). Since the interval $(a, b)_{\hat{C}}$ is linearly-ordered and $g(b) \in U$, we conclude $a \ast g(b) \ast b \ast fix_-(g)$, so that $\{b, g(b)\}$ is a cut-pair, and $X$ cannot be cut-rigid – a contradiction. ■

Figure 6: Diagrammatic representation of all possible interactions between $A, B \in \Xi_3$ satisfying $A \parallel B$. Note the bivalent cut pair arising in the rightmost diagram.
Case 2 – $A \cap B = \emptyset$: We first use the Bowditch tree to reduce the set of intervals $(A, B)$ that have to be considered. From the transversality lemma it follows that (in a cut-rigid space) no element of $\Theta$ is transverse to an element of $\Xi_3$; in addition, in [2] (lemma 3.17) it is proven that any set of elements of $\Theta$ separating $a$ from $b$ and bounded away from both $a$ and $b$ is necessarily finite. Thus, the set of $P \in \Theta$ satisfying $A \ast P \ast B$ is finite, and we may cut the interval $(A, B)$ into finitely-many subintervals of the form $(P, Q) := (P, Q)_{\Theta \cup \Xi_3}$, where the pair $\{P, Q\}$ forms a gap in $\Theta$; by proposition 5.5, there exists no cut pair cutting $X$ between $P$ and $Q$. Thus, theorem 3.14 may again be applied (this time directly to the space $X$ and the family $\mathcal{F} = \Xi_3$) to conclude that $(P, Q)$ is finite, and we are done. ■

To summarize, we have proved –

**Proposition 5.6** Suppose $X$ is a cut-rigid space admitting a cusp-uniform group action. Then the cut system $\Xi_3$ forms a discrete pretree with respect to the natural separation order. ■

### 5.2 Proof of the main theorem

Understanding that $\Xi_3$ is bivalent not only simplifies the resulting construction for its “enveloping” tree, but, in our particular case, provides new insights into the structure of this tree.

**Horocycles and stars**

Given a (necessarily parabolic) point $p$ in a cut $A_0 \in \Xi_3$, let us consider the set

$$H_p = \{ A \in \Xi_3 \mid p \in A \} .$$

We will call $H_p$ the horocycle through $\Xi_3$ centered at $p$. 

---

Figure 7: producing bivalent cut-pairs using a loxodromic transformation in a non-cut-rigid space.
The name chosen for $H_p$ requires some justification. Once again, let $X(p)$ be the end compactification of $X \setminus \{p\}$, with $\epsilon_\pm$ denoting the ends in this compactification. Using the pointed version of the natural separation order on the space $X(p)$, consider the family $H_p^*$ consisting of all $A^* = A \setminus \{p\}$ for $A \in H_p$, augmented by the univalent points $\epsilon_\pm$. By the definition of ends, the relation $\epsilon_- * A^* * \epsilon_+$ holds for all $A \in H_p$, meaning that $H_p^*$ is an interval, and so $H_p^*$ is linearly ordered by the natural separation order. Equivalently, this means $H_p$ is a linearly-ordered (up to a choice of orientation) subfamily of $\Xi_3$. Now consider the action of the parabolic subgroup $G_p$ on $X \setminus \{p\}$: since the action is co-compact and $H_p$ is discrete, we understand that $G_p$ acts co-finitely on $H_p$, and that $H_p$ is order-isomorphic to $\mathbb{Z}$. A visualization of this argument is given by the example of the reflection group of the Apollonian packing (or any other packing of finite type, actually) – see figure 2.

By construction, $\Xi_3$ is the union of all possible horocycles. Since the horocycles are indexed by parabolic points of $G$ and $g$ is cusp-uniform, we deduce that there are only finitely-many $G$-orbits of horocycles. The discreteness of horocycles then implies the following:

**Proposition 5.7** Suppose $G$ is any group acting cusp-uniformly and minimally on the cut-rigid space $X$. Then $G$ acts co-finitely on $\Xi_3$. ■

In particular, since $G_A$ is finite for all $A \in \Xi_3$, the order of $G_A$ is uniformly bounded by some constant depending on $G$ alone.

Let us return to the cut $A_0$ and the point $p \in A_0$. Since $H_p$ is order-isomorphic to $\mathbb{Z}$, it contains precisely two immediate neighbours of $A_0$ – call them $A_1$ and $A_{-1}$ – satisfying $A_{-1} * A_0 * A_1$. Since $\{A_0, A_1\}$ and $\{A_0, A_{-1}\}$ are null sets in $\Xi_3$, there exist unique stars $\sigma_+$ and $\sigma_-$ containing each of these sets, respectively. In order to prove that $\Xi_3$ is a bivalent pretree it will be enough to show there are no other stars containing $A_0$.

Let now $\sigma \subset \Xi_3$ be any star containing $A_0$, and take $A_0 \neq B \in \sigma$. Since $A_{-1} * A_0 * A_1$ holds, we have either $A_{-1} * A_0 * B$ or $B * A_0 * A_1$; without loss of generality assume the former.

If $A_1 \in \sigma$, then we are done: the intersection $\sigma \cap \sigma_+$ has more than one point, which implies $\sigma = \sigma_+$. Thus, we assume $A_1 \notin \sigma$, and let $G$ be its projection to $\sigma$ (recall that projections to stars are well-defined in a discrete pretree). Now, for any $A \in \sigma$, $A \neq G$ we have $A * G * A_1$. If $G$ is not $A_0$, then we have $A_0 * G * A_1$, which is impossible, because $A_0, A_1 \in H_p$ would have implied $G \in H_p$, but $A_0$ and $A_1$ are a gap in $H_p$ – contradiction. Thus, $G = A_0$ and we have $B * A_0 * A_1$.

Let us summarize: assuming $\sigma \neq \sigma_-, \sigma_+$ brought us to the conclusion that both the relations $A_{-1} * A_0 * B$ and $B * A_0 * A_1$ must hold for any $B \in \sigma$.
(B \notin H_p). This is impossible because the cut \( A_0 \) is bivalent: since \( \Xi_3 \) is nested, either \( B \subset C_{A_1,A_0} \) or \( B \subset C_{A_{-1},A_0} \). We have proven that \( \Xi_3 \) is a bivalent discrete pretree, and so proposition 2.8 realizes \( \Xi_3 \) as the edge-set of a tree \( \Xi_3 \), whose vertices are the stars of \( \Xi_3 \).

The discussion above may be repeated for each of the other points of \( A_0 \). Thus, a star in \( \Xi_3 \) may be specified by picking a horocycle \( H_p \) and a pair of neighbouring cuts \( A, B \in H_p \). The unique star containing this pair of cuts will be denoted by \( \sigma_{A,B} \). Note, hence, that every star intersects at least three different horocycles: for the general case, this means that \( \Xi_3 \) contains no gaps; the reader might also want to compare this to the example of the Apollonian packing, which is a minimal example in this sense, as every star in this packing intersects precisely three horocycles.

**Dynamics of cusp-uniform actions on \( \Xi_3 \)**

Knowing now that \( \Xi_3 \) is a well defined simplicial \( G \)-tree for *every* cusp-uniform \( G \) acting minimally on \( X \), the aim of this paragraph is to finish the proof of the main theorem.

Recall that the stabilizer of any \( A \in \Xi_3 \) is finite because \( G \) acts properly-discontinuously on \( 3X \). The same is not true for stabilizers of stars though, and we first prove –

**Proposition 5.8 (No global fixed points in \( \Xi_3 \))** For each vertex \( \sigma \) of \( \Xi_3 \), there exist a horocycle \( H_p \) and a parabolic element \( g \in G_p \) such that \( g\sigma \neq \sigma \).

**Proof:** Let \( \sigma \) be a star in \( \Xi_3 \), and we claim there is a horocycle \( H_p \) such that some \( g \in G_p \) does not stabilize \( \sigma \). This is already enough for proving that \( \sigma \) is not \( G \)-stable, but we also have the assumption that \( p \) is a bivalent bounded parabolic point, so proposition 4.4 implies \( G_p \) is a 2-ended group. Choosing \( g \) more accurately so that it is a parabolic element virtually generating \( G_p \), we will prove the more subtle assertion of the proposition.

Let now \( A, B \) be an adjacent pair of elements in a horocycle \( H_p \) intersecting \( \sigma \). Use the reasoning above to produce a parabolic transformation sending the pair \( \{A,B\} \) to some disjoint pair (think of \( g \) as a translation along \( H_p \)). We then have –

\[
g \cdot \sigma = g \cdot \sigma_{A,B} = \sigma_{gA,gB};
\]

Since \( \sigma \cap H_p = \{A, B\} \), we conclude that \( g\sigma \neq \sigma \). ■

We are interested in conditions on the finiteness of star-stabilizers. The fact that \( G \) acts co-finitely on \( \Xi_3 \) implies the following –
Proposition 5.9  Let $\sigma$ be a star in $\Xi_3$. Then, $G_\sigma$ acts co-finitely on $\sigma$. In particular, $G_\sigma$ is finite if and only if $\sigma$ is finite. 

We have thus shown $G$ to be the fundamental group of a finite graph with finite edge-groups and possibly-infinite vertex groups. The parabolic subgroups of $G$ fixing points which are contained in elements of $\Theta$ translate edges within horocycles, thus producing splittings over finite subgroups.

It is interesting to understand the role played by the idle subgroups in this presentation of $G$, but our exposition here will be limited to understanding the placing of these subgroups of $G$ with respect to the splittings arising from the action of $G$ on $\Xi_3$:

Lemma 5.10 (translations of $\Xi_3$)  Suppose $g \in G$ translates an edge $e$ of $\Xi_3$. Then either $g \in G_p$ for some parabolic point $p$ and $e \in H_p \Theta$, or $g$ is a loxodromic element of $G$.

Proof:  given a star $\sigma$ and $A \in \sigma$, let us denote by $\text{int}_\sigma(A)$ the unique component of $X \setminus A$ containing all elements $B \in \sigma \setminus \{A\}$; the other component of $X \setminus A$ will be denoted by $\text{ext}_\sigma(A)$.

Suppose $g$ does not translate $e$ within any horocycle. Write $e = (A, \sigma)$, and then we know that $e$ points at $g \cdot e = (gA, g\sigma)$, while $g \cdot e$ points away from $e$. This means that $A \ast \sigma \ast gA$ and $\sigma \ast gA \ast g\sigma$ both hold in $\Xi_3$, which, in turn, means that $K = \text{int}_\sigma(A)$ contains $gK = \text{int}_{g\sigma}(gA)$. Now, since $A$ and $gA$ have no point in common (or else $g$ translates $e$ within the horocycle determined by the point of intersection), we have $gK$ lying in $\text{int}(K)$, which proves that $g$ determines a loxodromic transformation of $X$. ■

Note:  It is interesting to note that not every splitting of $G$ over a virtually-$\mathbb{Z}$ subgroup is induced from a translation of $\Xi_3$, as every multivalent cut pair $P$ determines an infinite star stabilized by $G_P$.

Finally, we are only left to prove the minimality of $\Xi_3$ as a $G$-tree. Suppose $A, B \in \Theta$ are distinct, let $\sigma, \tau$ be stars containing $A$ and $B$ respectively and lying on the path connecting $A$ to $B$. Recall that for any discrete convergence group it is true that the set of loxodromic pairs is dense in $2X$. Therefore, there exists a loxodromic $g \in G$ with $fix_-(g) \in \text{ext}_\sigma(A)$ and $fix_+(g) \in \text{ext}_\tau(B)$, so that $g^n(A) \subset \text{ext}_\tau(B)$ for some sufficiently large $n$. Thus, $B$ lies in the subtree of $\Xi_3$ spanned by the $G$-orbit of $A$, and we have completed the proof of the main theorem.
6 Questions and discussion

All the questions regarding $Ξ_3$ arise naturally as the proof of the main theorem proceeds:

**Rigidity of $X$:** It would be interesting, as well as useful, to get rid of the rigidity assumption, for then it might be possible to understand the more subtle splittings of a cusp-uniform group $G$ the same way it was done in [2] for uniform groups.

Note that, in the cut-rigid case, the pretree of multivalent exact cut pairs may be unified with the pretree $Ξ_3$ to produce a discrete invariant tree: adding elements of $Θ$ to $Ξ_3$ splits some of the stars of $Ξ_3$ into smaller stars, thus providing us not only with information regarding splittings of $G$ over finite subgroups, but also with information regarding splittings of $G$ over two-ended groups.

**Discreteness of $Ξ_3$:** It seems that the discreteness of $Ξ_3$ is a result of a very delicate interplay between $Ξ_3$ and the Bowditch tree. Is there a way to extend our results to provide an understanding of exact cuts of higher orders in some discrete combinatorial setting? – the guess is that such an understanding may result in a converse to our Theorem A:

**Question 1:** Suppose $G$ acts cusp-uniformly and minimally on a cut-rigid space $X$. If $G$ has more than one end, does $G$ stabilize an exact cut in $X$?

**Rigidity of $G$:** Further study of the structure of $Ξ_3$ hints at the possibility that the combinatorial restrictions on $Ξ_3$ arising from the sole existence of a cusp-uniform group acting minimally on $X$ might be such that the question above may be strengthened to the form:

**Question 2:** Suppose $G$ acts cusp-uniformly and minimally on a cut-rigid space $X$. If $X$ contains a bivalent cut-triple, is then $G$ virtually free?

In this context, it is important to discuss another possible approach to proving $G$ has more than one end. Note that this approach may be better than ours for this particular purpose, because it succeeds in considering *all* the bivalent parabolic points of $G$, including the idle ones. The only drawback of this approach in our view is that, unlike
our own, it does not supply one with a *canonical* tree describing the corresponding splitting for *any* group $G$.

Given a cut-rigid space $X$ and a cusp-uniform action by a group $G$, one regards $G$ as a group (strongly) hyperbolic relative to the collection $\mathcal{P}$ of stabilizers of parabolic points. Our proposition 4.4 then shows the stabilizers of bivalent parabolic points are two-ended, implying $G$ is hyperbolic relative to the subcollection $\mathcal{P}'$ of parabolic points of valence unequal to 2, by corollary 1.14 in the paper [8] by Drutu and Sapir (at least under the additional assumption that all parabolic groups are finitely generated). The pair $(G, \mathcal{P}')$ will then have a new (uniquely defined) boundary $X'$, by the results of [18], and it is possible to show, using an argument by Dahmani ([10], proof of lemma 4.4), that $X'$ admits a $G$-equivariant continuous map onto $X$, which is two-to-one on the preimage of bivalent parabolic points and one-to-one on its complement (essentially, one could think of $X'$ as a result of “blowing $X$ up” simultaneously at all the parabolic points of valence 2). It then involves a simple combinatorial argument to show that $X'$ is disconnected if it contains an exact cut triple, implying $G$ has more than one end.

The last stage of the proof is the particular stage where information regarding splittings seems to be lost. In fact, a “black box” argument is used in order to say $G$ has more than one end: since we deduce this information from $X'$ being disconnected, it is unclear what tree should one use in order to describe the splittings of $G$ corresponding to the fact it has more than one end. This suggests that the information lost in the “blowing up” process (when $X'$ is constructed) is precisely the kind of information we have begun to analyze in this paper: the combinatorics of bivalent cut sets in $X$.

**Rigidity of $X$:** Because of the canonicity of our construction, it seems plausible to formulate the last question in an even stronger form:

**Question 3:** Suppose $X$ is a cut-rigid space admitting a minimal cusp-uniform group action. If $X$ contains an exact cut triple, is it true that $X$ is a combination (for example, in the sense of [10]) of tight circle packings of finite type?

A more immediate goal in this respect would be to answer:

**Question 4:** Suppose $X$ is a cut-rigid space admitting a minimal cusp-uniform group action. If $X$ contains an exact cut triple, and all stars in $\Xi_3$ are finite, is it true that:
1. there exists a homeomorphism $h$ of $X$ onto a circle-packing $P$ of finite type?

2. every cusp-uniform group $G$ of homeomorphisms of $X$ is conjugate (via $h$) into the stabilizer of $P$ in the isometry group of hyperbolic space? (recall from [18] that $X$ carries a $G$-invariant hyperbolic crossratio; this question resembles asking whether or not this crossratio is the same for every choice of $G$)

In attempt to provide affirmative answers to the above questions, the author has undertaken a deeper study of the structure of stars in $\Xi_3$. This study resulted in finding interesting examples of circle-packings not admitting any cusp-uniform group action, as well as more fine-tuned versions of the main theorem of this paper. We intend to give an account of these results in a future paper.

References


