Bordifications of cubings

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Cubings

Definition: A cubing is a simply-connected non-positively curved cube complex. The piecewise Euclidean path pseudometric on a cubing $X$ is a complete CAT(0) metric (by Bridson's theorem) if either –

1. $X$ is finite-dimensional, or
2. $X$ is locally finite.

As a result, in both cases $X$ has different bordications attached to it, e.g.:

Visual boundaries: cone, Tits, ne;

Combinatorial boundaries: cube, Roller.
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- **Ballman-Buyalo:** If a Tits boundary of a CAT(0) group $G$ has diameter greater than $2\pi$ then $G$ has rank-1 elements.

- **Swenson-Papasoglu:** Improved the bound to $3\pi/2$, and much more.
Hyperplanes

Theorem (Sageev):

A hyperplane \( W \) of a cubing \( X \) does not self-intersect, is closed and convex, is itself a cubing, separates \( X \) into two convex components with common boundary \( W \).
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Halfspace system

Definition:
The halfspace system $H(X)$ of a cubing $X$ is the set of all complementary components of hyperplanes (halfspaces) in $X$. We also throw in the empty set and $X$.

$H(X)$ is viewed as a poset (w.r.t. inclusion) endowed with the order-reversing involution $a \mapsto a_{\mathcal{A}}$ where $a_{\mathcal{A}} = X - a$.

Definition (Roller):
$H$ is said to be $\mathcal{A}$-dimensional, if $H$ contains no infinite transverse family of halfspaces.

From now on we only work with $\mathcal{A}$-dimensional cubings.
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Reconstruction (Sageev-Roller)
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- Definition (Roller):

A family $H$ is coherent, if no $a, b \in s$ satisfy $a \leq b$. A maximal coherent family is an ultralter.

Denote the set of all ultralters by $H$. Construct $X_0 \neq H$ by $v \neq (v')$ where $(v') = \{a \in H | v \leq a\}$. Im($\ll$)s an almost-equality class in $H$ – the unique almost-equality class of all ultralters not containing infinite descending chains. We call this class the principal class.
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- **Definition (Roller):** A family $\alpha \subset H$ is *coherent*, if no $a, b \in \alpha$ satisfy $a \leq b^*$. 
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- **Construct** $X^0 \to H^\circ$ by $\nu \mapsto \pi(\nu)$ where

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Reconstruction (cntd.)

Construct a cube complex as follows: vertex set: \( H \), edge set: \( \{x \neq y \mid \text{d}(x, y) \leq 1\} \). Glue one Euclidean square to each \( 4 \)-cycle.

Continue inductively, attaching \([0, 1]^d\) to each instance of \( @([0, 1]^d) \).

Theorem (Sageev): The resulting cube complex \( C(H) \) is the disjoint union of cubings. The principal component is naturally isomorphic to \( X \).
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**Theorem (G.):** $\mathcal{R}H$ is a meet semilattice with respect to the partial ordering defined by

$$A \leq B \iff B \subseteq \bar{A} \iff B \cap \bar{A} \neq \emptyset.$$
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Moreover, there is a natural system of commuting projections $\{ A \to B \}_{A \leq B}$ continuously extending to the corresponding closures.
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Boundary decomposition map
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- **Theorem (G.):** Suppose $X$ is co-compact. The function $\rho : \partial_\infty X \to \mathcal{R}H$ defined by setting $\rho(\xi)$ to equal the smallest element of $\mathcal{R}H$ containing the limit of a sequence of vertices of $X$ converging to $\xi$, is well defined and

$$\rho^{-1}(A) = \bigcup_{B \leq A} \rho^{-1}(B)$$

for all $A \in \text{Im}(\rho)$. 
Boundary decomposition map

- **Theorem (G.):** Suppose $X$ is co-compact. The function $\rho : \partial_\infty X \to RH$ defined by setting $\rho(\xi)$ to equal the smallest element of $RH$ containing the limit of a sequence of vertices of $X$ converging to $\xi$, is well defined and

$$\rho^{-1}(A) = \bigcup_{B \leq A} \rho^{-1}(B)$$

for all $A \in \text{Im}(\rho)$.

- **Note:** For a general cubing $\rho$ is not necessarily surjective.
Suppose $X$ admits a geometric action by a group $G$. Is surjective? A positive answer will imply, for example, that a finitely generated subgroup of $G$ is virtually abelian iff the image of its limit set in the Roller boundary is finite.

Theorem (G.): Suppose the cubing $X$ is co-compact. Let $\mathcal{C}$ be the comparability graph on $\text{Im}(\mathcal{C})$. Then the components of the Tits boundary $\partial T_X$ correspond to components of $\mathcal{C}$ under $\mathcal{C}$.

Remark: Together with a positive answer to the above, this gets us closer to the goal of studying splittings over virtually abelian subgroups using the CAT(0) boundary: look for invariant nested families of finite cuts in $X$.
**Boundary decomposition map**

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• **Remark:** Together with a positive answer to the above, this gets us closer to the goal of studying splittings over virtually abelian subgroups using the CAT(0) boundary: look for invariant nested families of finite cuts in $\Gamma$. 
Theorem (Roller):
The family of $1$-skeleta of cubings is precisely the family of median graphs.

Intervals defined by pairs of vertices in a graph:
$I(u; v) = \{t \mid d(u; t) + d(t; v) = d(u; v)\}$.

A median graph is a graph in which $I(u; v) \setminus I(v; w) \setminus I(u; w)$ is a point, for all vertices $u; v; w$. 
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  is a point, for all vertices \( u, v, w \).
Median operation

\[ \text{med}(u; v; w) = (u \lor v) \land (u \lor w) \land (v \lor w) \]
Median operation

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  \text{med}(u, v, w) = (u \cap v) \cup (u \cap w) \cup (v \cap w).
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- $C'(H)^1$ has a median operation:
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Write $p = (a, b)$, $p_0 = (0, 0)$ and $q = (x, y)$. If $a \gg x$ and $b \gg y$,

$$f_p(q) = |a - x| + |b - y| - |a| - |b|$$

$$= -x - y,$$

which is independent of $a$ and $b$. 
Example (2)

Consider the plane with the median metric ($\ell_1$ metric), fix a point $p_0$, and construct the metric bordification:

$$p \mapsto f_p(\cdot) = d(\cdot, p) - d(p, p_0).$$

Write $p = (a, b)$, $p_0 = (0, 0)$ and $q = (x, y)$. If $a \gg x$ and $y \sim b$,

$$f_p(q) = |a - x| + |b - y| - |a| - |b|$$

$$= -x + (|b - y| - |b|),$$

which becomes independent of $b$ if considered modulo bounded functions!
More elegant description? (2)
Theorem (Bader-G.): The metric compactification of the 1-skeleton of a cubing coincides with its cube boundary compactification. The identification map is a bi-Lipschitz homeomorphism putting boundedness classes in one-to-one correspondence with almost-equality classes. In particular, the Roller boundary of $X$ is the quotient of the metric compactification of $X^1$ by bounded functions.
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