TOPOLOGICAL PROPERTIES OF REFLECTIONLESS JACOBI MATRICES

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ABSTRACT. We study the topological properties of spaces of reflectionless Jacobi matrices.

1. Introduction

This paper is part of my attempts to achieve a better understanding of the absolutely continuous spectrum of Jacobi matrices and, in particular, reflectionless operators. This program was begun in [12]; please also see [9, 10, 11] for subsequent work along these lines.

A *Jacobi matrix* is a difference operator of the following type:

$$(Ju)_n = a_n u_{n+1} + a_{n-1} u_{n-1} + b_n u_n$$

Here, $a_n \geq 0$ and $b_n \in \mathbb{R}$, and we also always assume that a, b are bounded sequences.

Alternatively, one can represent J by a tridiagonal matrix with respect to the standard basis of $\ell^2(\mathbb{Z})$:

$$J = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & a_{-2} & b_{-1} & a_{-1} & & & \\ & & a_{-1} & b_0 & a_0 & & \\ & & & a_0 & b_1 & a_1 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}$$

In this paper, we will study spaces of Jacobi matrices, not individual operators. We will write

$$\mathcal{J}_R = \{ J : \|J\| \leq R \}$$

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for the set of Jacobi matrices satisfying a given bound on their operator norms. On these spaces, we then use the following metric:

$$d(J,J') = \sum_{n=-\infty}^{\infty} 2^{-|n|} \left(|a_n - a'_n| + |b_n - b'_n| \right) \tag{1.1}$$

This is a natural choice; for example, the topology induced by d on $\mathcal{J}_R$ coincides with the weak and strong operator topologies, and, as we will see, d interacts well with natural modes of convergence of other quantities. Moreover, $(\mathcal{J}_R, d)$ is a compact space.

A Jacobi matrix J is called *reflectionless* on a Borel set $M \subset \mathbb{R}$ if

$$\text{Re } g_n(t) = 0 \quad \text{for Lebesgue almost every } t \in M$$

for all $n \in \mathbb{Z}$. Here, $g_n(z) = \langle \delta_n, (J-z)^{-1}\delta_n \rangle$ is the Green function of J at site n, and $g_n(t) = \lim_{y\to 0+} g_n(t+iy)$; this limit exists for almost every $t \in \mathbb{R}$. See [12] for more information on reflectionless operators and why they are important.

The set of reflectionless (on M) Jacobi matrices will be denoted by $\mathcal{R}(M)$. We will be particularly interested in the subspaces

$$\mathcal{R}_0(K) = \{ J \in \mathcal{R}(K) : \sigma(J) \subset K \}$$

for compact sets $K \subset \mathbb{R}$. Here, $\sigma(J)$ denotes the spectrum of J; if K is an essentially closed set (that is, it intersects an arbitrary open interval either in a set of positive Lebesgue measure or not at all), then in fact $\sigma(J) = K$ for all $J \in \mathcal{R}_0(K)$.

Throughout this paper, we make the following

basic assumption: The set K is of positive Lebesgue measure.

The case $|K| = 0$ is of no interest at all in this context because then the condition of being reflectionless on K becomes vacuous. More to the point, we avoid awkward supplements to our statements if we assume that $|K| > 0$. See also the final section of [10] for a discussion of these issues.

The spaces $\mathcal{R}_0(K)$ are themselves compact if endowed with the metric d from (1.1). Moreover, the shift S acts as a homeomorphism on $\mathcal{R}_0(K)$, so we obtain (topological) dynamical systems $(\mathcal{R}_0(K), S)$. Here, we define the action of S on a Jacobi matrix J in the obvious way, by shifting its coefficients a_n, b_n.

These dynamical systems $(\mathcal{R}_0(K), S)$ are of considerable interest. For instance, they provide natural examples of systems with absolutely continuous spectrum and are thus relevant to the Kotani-Last question. For more background information on these topics, see [3, 5].

For simple choices of K, the spaces $\mathcal{R}_0(K)$ consist of well known classes of Jacobi matrices. If K is a single interval, then $\mathcal{R}_0(K)$ only has one element, and this Jacobi matrix has constant coefficients; if specifically $K = [-2, 2]$, then we obtain free Jacobi matrix ($a_n = 1$, $b_n = 0$). If we add isolated points, that is,

$$K = [-2, 2] \cup \{x_1, \dots, x_N\},$$

then the members of $\mathcal{R}_0(K)$ are usually referred to as *solitons* (with eigenvalues given by a subset of $\{x_1, \dots, x_N\}$, possibly the empty set). If K is a *finite gap set* (a disjoint union of finitely many compact intervals of positive length), then $\mathcal{R}_0(K)$ contains exactly the finite gap Jacobi matrices with spectrum K.

The dynamics of $(\mathcal{R}_0(K), S)$ are very well understood in these cases. For example, if K is a finite gap set whose complement $\mathbb{R} \setminus K$ has N bounded components (referred to as *gaps* in the sequel), then $(\mathcal{R}_0(K), S)$ is isomorphic to $(\mathbb{T}^N, T_a)$, a translation on the N-dimensional torus; in fact, rather explicit information is available on both the isomorphism and a from T_a. These finite gap systems have been studied very extensively.

More recent work by Sodin and Yuditskii [13] establishes the analogous result for certain sets K with infinitely many gaps. For *general* sets K with infinitely many gaps, not much is known at present.

In this paper, I will make no attempt to study $\mathcal{R}_0(K)$ as a dynamical system. Instead, I will settle for the more modest assignment of trying to understand $\mathcal{R}_0(K)$ as a topological space but I will make a point of attacking this question for completely general sets K.

Our basic tool will be a representation of $\mathcal{R}_0(K)$ as a fibered space over a (usually: infinite-dimensional) torus.

It is perhaps useful to state this more formally. As above, let $N \in \mathbb{N}_0 \cup \{\infty\}$ be the number of bounded components *(gaps)* of $\mathbb{R} \setminus K$. Denote the torus of this dimension by $\mathcal{T} = \mathbb{T}^N$; if $N = \infty$, then this is defined as the infinite Cartesian product $\mathbb{T}^\infty = S^{\mathbb{N}}$ of countably many copies of the unit circle S, endowed with the product topology. Notice that $\mathcal{T}$ is a compact metrizable space for arbitrary $N \in \mathbb{N}_0 \cup \{\infty\}$.

Proposition 1.1. *There exists a continuous surjective map*

$$p : \mathcal{R}_0(K) \to \mathcal{T}.$$

We then call $p^{-1}(\{t\})$ the *fiber* over $t \in \mathcal{T}$. The formulation of the Proposition may in fact be slightly misleading in that abstract existence of p is not really the main point here; rather, we will always work with a concrete, natural choice for p, which will be properly defined in Section 2.

Such a representation of $\mathcal{R}_0(K)$ has been used by many authors; it appears that these ideas go back to [2], but see also [6, 14]. It also played a central role in [10, 12], even though somewhat different terminology was used there (fibered spaces weren't mentioned). We'll review this material in Section 2.

It frequently happens that all fibers are single points, so that $\mathcal{R}_0(K)$ is then homeomorphic to $\mathcal{T}$. In general, the following can be said.

Theorem 1.2. *The map p from Proposition 1.1 can be chosen so that it has the following additional property: Let*

$$\delta(t) = \operatorname{diam}\left(p^{-1}(\{t\})\right) = \max\{d(J, J') : p(J) = p(J') = t\}.$$

Then δ is an upper semicontinuous function on $\mathcal{T}$. Moreover, $\{t \in \mathcal{T} : \delta(t) = 0\}$ is a dense G_δ set in $\mathcal{T}$.

Fibers may vary wildly, however, and δ is certainly not continuous in general. It is easy to build examples where the complement $\{t : \delta(t) > 0\}$ is also dense in $\mathcal{T}$. A good toy model to keep in mind is the topologist's sine curve

$$\{(t, \sin(1/t)) : 0 < t \leq \pi\} \cup \{(0, y) : -1 \leq y \leq 1\},$$

viewed as a fibered space over $[0, \pi]$ in the natural way, with p being the projection onto the x-axis. (This has only one non-trivial fiber, but of course a simple modification will give a space with non-trivial fibers on a dense set.)

This model must not be taken too seriously. Indeed, unlike the topologist's sine curve, the spaces $\mathcal{R}_0(K)$ satisfy:

Theorem 1.3. *$\mathcal{R}_0(K)$ is pathwise connected.*

This has the following consequence.

Corollary 1.4. *If $\mathcal{R}_0(K)$ can be made an abelian topological group, then $\mathcal{R}_0(K)$ is homeomorphic to $\mathcal{T}$.*

See [4, Theorem 8.46] for this conclusion. The Corollary seems to be of some interest here because, as is well known, such a group structure can be obtained from the dynamics if $(\mathcal{R}_0(K), S)$ is a minimal system containing only Jacobi matrices with almost periodic coefficients: one just extends to all of $\mathcal{R}_0(K)$ the group structure on an orbit that is provided by the natural action of $\mathbb{Z}$.

If there are non-trivial fibers, then their effect on the topology of $\mathcal{R}_0(K)$ is not easy to understand. The simplest situation where such non-trivial fibers are present occurs when K has isolated points, and here the simplest example is provided by sets of the type

$$K = [-2, -1] \cup \{0\} \cup [1, 2], \tag{1.2}$$

say. This K has two gaps, thus $\mathcal{T} = \mathbb{T}^2$, and it will also be easy to see that there is exactly one point $t_0 \in \mathcal{T}$ with a non-trivial fiber; this fiber is homeomorphic to a closed interval. Superficially, just given this information, it is not even clear if $\mathcal{R}_0(K)$ is still a topological manifold (locally homeomorphic to $\mathbb{R}^2$).

The effect of such isolated points of K on the topology of $\mathcal{R}_0(K)$ is clarified by Theorem 1.5 below. In fact, they have no effect at all: $\mathcal{R}_0(K) \cong \mathbb{T}^2$ for the K from (1.2). To formulate such a statement in precise terms in a general setting, suppose that $x_0 \in K$ is an isolated point of a general compact set K, and consider the modified sets

$$K_t = K \cup [x_0 - t, x_0 + t],$$

where $t > 0$ is chosen so small that $[x_0 - t, x_0 + t] \cap K = \{x_0\}$.

Theorem 1.5. *$\mathcal{R}_0(K)$ and $\mathcal{R}_0(K_t)$ for $t > 0$ are homeomorphic.*

This addresses the issue we just discussed because that part of the non-trivial fiber of $\mathcal{R}_0(K)$ that can be said to correspond to x_0 disappears when we consider $\mathcal{R}_0(K_t)$ for $t > 0$. (We spoke of a fiber of $\mathcal{R}_0(K)$, for convenience, but of course fibers are associated with the map $p : \mathcal{R}_0(K) \to \mathcal{T}$ and they are not an intrinsic property of the space $\mathcal{R}_0(K)$.) See again Section 2 and also Lemma 4.1 for more details on this.

As a by-product, Theorem 1.5 also clarifies the nature of spaces of soliton type Jacobi matrices.

Corollary 1.6. *Suppose that*

$$K = \bigcup_{n=1}^{N+1} I_n$$

is a disjoint union of $N+1$ compact intervals $I_n = [a_n, a_n + d_n]$, $d_n \geq 0$, some of which may be single points. Then $\mathcal{R}_0(K)$ is homeomorphic to $\mathbb{T}^N$.

Proof. This is well known if all intervals are of positive length. The general case now follows from Theorem 1.5. □

In particular, if we take $K = [-2, 2] \,\dot{\cup}\, \{x_1, \ldots, x_N\}$, so that $\mathcal{R}_0(K)$ becomes the spaces of all (classical) solitons whose eigenvalues are a subcollection of the points $x_1, \ldots, x_N$, then $\mathcal{R}_0(K) \cong \mathbb{T}^N$.

So non-trivial fibers need not have any effect on the topology of $\mathcal{R}_0(K)$. However, are there examples where something of this sort happens? I do not know this, and in fact I am unable to answer, at this point, the following very basic sounding

Open question: Are there any (compact, positive measure) sets $K \subset \mathbb{R}$ for which $\mathcal{R}_0(K)$ is *not* homeomorphic to a torus?

I believe that these issues deserve further study.

2. $\mathcal{R}_0(K)$ as a fibered space

In this section, we review material that was discussed in more detail in [10, 12]. Please refer to these sources for additional information.

Given a reflectionless Jacobi matrix $J \in \mathcal{R}_0(K)$, consider its H function $H(z) = -1/g_0(z)$ and the associated Krein function

$$\xi(t) = \frac{1}{\pi} \lim_{y\to 0+} \text{Im } \ln H(t+iy).$$

Notice, first of all, that Im $H(z) > 0$ on the upper half plane $z \in \mathbb{C}^+ = \{z = x+iy : y > 0\}$, so H has a holomorphic logarithm on $\mathbb{C}^+$ and we may demand that Im $\ln H \in (0,\pi)$ there. By general facts about Herglotz functions, we now see that the limit defining $\xi(t)$ exists for Lebesgue almost every $t \in \mathbb{R}$, and $0 \le \xi \le 1$. If viewed as an element of $L^\infty(\mathbb{R})$, then ξ is uniquely determined by H, and, conversely, we can recover H from ξ. The following explicit formula holds:

$$H(z) = (z+R)\exp\left(\int_{-R}^{R} \frac{\xi(t)\, dt}{t-z} \right); \tag{2.1}$$

here, $R > 0$ must be chosen so large that $\|J\| \le R$, but is otherwise arbitrary.

The fact that $J \in \mathcal{R}_0(K)$ implies that $\xi = 1/2$ almost everywhere on K. This follows because g_0 and thus also H are purely imaginary on this set. Moreover, it is not hard to show (please see [10, 12] for details) that $\xi = 1$ to the left of K and $\xi = 0$ to the right of K, and on each gap $(a,b) \subset K^c$, the Krein function jumps from 0 to 1 (or not at all), that is, its restriction to this gap is of the form $\xi = \chi_{(\mu,b)}$ for some $\mu \in [a,b]$.

In other words, the ξ function of a $J \in \mathcal{R}_0(K)$ is uniquely determined by the collection of its jump points (μ_j) (one for each gap). Conversely, if $\mu_j \in [a_j, b_j]$ are given, then there exist Jacobi matrices $J \in \mathcal{R}_0(K)$ whose ξ functions have these parameters.

These ξ functions do not, in general, provide a complete parametrization of $\mathcal{R}_0(K)$; in fact, unless K is a single interval, there are always distinct Jacobi matrices $J \in \mathcal{R}_0(K)$ that share the same ξ.

To obtain a bijective map from $\mathcal{R}_0(K)$ onto an enlarged set of spectral data, we write down the Herglotz representation of $H(z)$. This will take the form

$$H(z) = z + A + \int_{(-R,R)} \frac{d\rho(t)}{t-z}, \tag{2.2}$$

and here $A \in \mathbb{R}$ and the finite, positive Borel measure ρ are of course uniquely determined by ξ (since H is). More can be said about ρ. This measure is of the form

$$d\rho = \chi_K \, d\rho + \sum w_j \delta_{\mu_j}; \tag{2.3}$$

the sum is over those gaps for which $\mu_j \neq a_j, b_j$, and $w_j > 0$ for these j.

The significance of these new data lies in the fact that

$$\rho = \nu_+ + \nu_-$$

is the sum of the (suitably defined, see [10] for details) half line spectral measures $\nu_\pm$. Moreover, and this is the key point, the ν_+ obtained in this way are in one-to-one correspondence to the Jacobi matrices $J \in \mathcal{R}_0(K)$. In particular, we are claiming here that $J \in \mathcal{R}_0(K)$ can be reconstructed from just one half line spectral measure; this is one way of expressing the well known fact that reflectionless Jacobi matrices are already determined if their coefficients are known on an arbitrary half line.

The correspondence outlined above can be made much more explicit. The following two facts are crucial:

(1) Every ν_+ obtained in this way from a $J \in \mathcal{R}_0(K)$ is of the form

$$d\nu_+ = \frac{1}{2}\left(d\rho + g\chi_K \, d\rho_s + \sum \sigma_j w_j \delta_{\mu_j} \right), \tag{2.4}$$

with $g \in L^\infty(K)$, $-1 \le g \le 1$, and $\sigma_j \in \{-1, 1\}$. Also, as explained, $d\rho$ and $w_j = \rho(\{\mu_j\})$ are determined by the μ_j (because H and ξ are).

(2) Conversely, suppose that such data are given. In other words, suppose that $\mu_j \in [a_j, b_j]$, $\sigma_j \in \{-1, 1\}$ (for those j for which $\mu_j \neq a_j, b_j$), and $g : K \to [-1, 1]$ is a Borel function. Then, as discussed above, the μ_j determine a ξ function which, in turn, yields a unique $H(z)$ via (2.1). Use this H to obtain a unique Borel measure ρ from (2.2). This ρ will be of the form (2.3), and thus we may finally define ν_+ by (2.4). Then there exists a unique $J \in \mathcal{R}_0(K)$ whose ν_+ is given by this measure.

Given these observations, it is now convenient to think of this parametrization as a bijection $J \leftrightarrow (\mu, \sigma, g(x))$. Here we identify data (μ, σ, g), (μ', σ', g') that lead to the same ν_+. This happens if and only if $\mu_j = \mu'_j$ for all j and $\sigma_j = \sigma'_j$ for all j with $\mu_j \neq a_j, b_j$ and

$g = g'$ almost everywhere on K with respect to ρ_s. (In particular, if $\rho_s(K) = 0$, as happens quite frequently, then g becomes irrelevant.)

We equip this collection of data (μ, σ, g) with a metric that corresponds to the weak-$*$ topology for the ν_+. More precisely, we fix a metric D_0 that induces the weak-$*$ topology on the positive Borel measures λ on $[-R, R]$ with $\lambda([-R, R]) \le C$, and put

$$D((\mu, \sigma, g), (\mu', \sigma', g')) = D_0(\nu_+, \nu'_+), \tag{2.5}$$

where the measures ν_+, ν'_+ are constructed from the given data as described above.

This representation of $\mathcal{R}_0(K)$ provides a homeomorphism onto a suitably defined space $Y = Y(K)$ of spectral data $y = (\mu, \sigma, g)$. More precisely, let $Y(K)$ be the set of equivalence classes of such data $y = (\mu, \sigma, g)$. In other words, $y = (\mu, \sigma, g)$ is admissible as a set of data here if $\mu_j \in [a_j, b_j]$ and we are given a $\sigma_j = \pm 1$ for each j with $\mu_j \ne a_j, b_j$, and also a Borel function $-1 \le g \le 1$ on K. Two such data y, y' are identified (they are equivalent) if they lead to the same ν_+, or, what is the same, if $D(y, y') = 0$.

Proposition 2.1. *This map $(\mathcal{R}_0(K), d) \to (Y(K), D)$ is a homeomorphism between compact metric spaces.*

This was formulated as Proposition 2.5 of [10]; please see this reference for the straightforward proof.

Why did we refer to this representation of $\mathcal{R}_0(K)$ as a fibered space over $\mathcal{T}$? This comes from identifying $\widehat{\mu}_j \equiv (\mu_j, \sigma_j)$ for fixed j with a copy of the unit circle in the obvious way. Loosely speaking, just glue together the two copies of the interval $\mu_j \in (a_j, b_j)$ that correspond to the choices $\sigma_j = -1$ and $\sigma_j = 1$, respectively, at the points $\mu_j = a_j, b_j$.

If a more formal definition is desired, we can proceed as follows: Let f_j be the map that sends

$$(\mu_j, \sigma_j) \mapsto e^{i\pi\sigma_j(b_j - \mu_j)/(b_j - a_j)},$$

and also $f_j(b_j) = 1$, $f_j(a_j) = -1$, and define $f : Y(K) \to \mathcal{T}$ by applying these maps componentwise, that is, $f(y) = (f_j(\widehat{\mu}_j))_j$.

Proposition 2.2. *The map $p : \mathcal{R}_0(K) \to \mathcal{T}$, $J \mapsto y \mapsto f(y)$, is continuous and surjective. Moreover, a fiber $p^{-1}(\{f(y)\})$ consists of a single point precisely if $\rho_s(K) = 0$ for the ρ determined by μ.*

Sketch of proof. It is obvious from the construction of p that p is onto. The continuity of p is established in [10], in the first part of the proof of [10, Theorem 1.5]. The final claim is obvious from the definition of p and the previous discussion. $\square$

It will often be convenient to ignore the identification provided by f, and thus we will frequently think of p as the map that sends J to $\widehat{\mu}$, thought of as an element of $\mathcal{T}$.

In the sequel, we will need to know how to obtain detailed information about ρ from the boundary behavior of $H(t+iy)$ as $y \to 0+$. As usual, we write $H(t) \equiv \lim_{y\to 0+} H(t+iy)$; we know that this limit exists for Lebesgue almost every $t \in \mathbb{R}$.

Lemma 2.3. *(a)*

$$d\rho_{ac}(t) = \frac{1}{\pi}\, \chi_K(t)|H(t)|\, dt$$

(b) Let $d\rho_1 = f\, d\rho_{2,s} + d\sigma$ *be the Lebesgue decomposition of* ρ_1 *with respect to the singular part of* ρ_2 *(so* $\sigma \perp \rho_{2,s}$*). Then*

$$\lim_{y\to 0+} \frac{H_1(t+iy)}{H_2(t+iy)} = f(t)$$

for $\rho_{2,s}$*-almost every* $t \in \mathbb{R}$*.*

Proof. In general, we know that if μ is the measure from the Herglotz representation of a Herglotz function F, then $\pi d\mu_{ac}(t) = \operatorname{Im} F(t)\, dt$. This gives part (a) because $\operatorname{Re} H = 0$ on K and $\operatorname{Im} H = 0$ on the complement $\mathbb{R} \setminus K$.

Part (b) is one version of Poltoratski's theorem on the comparison of the singular parts from [8]; see also [1] for further background. □

These quantities can be conveniently estimated with the help of the Hilbert transform. We will make use of its truncated version

$$(T_y f)(x) = \int_{|t-x|>y;|t|\le R} \frac{f(t)\, dt}{t-x} \tag{2.6}$$

as well as the full transform

$$(Tf)(x) = \lim_{y\to 0+} (T_y f)(x);$$

this limit exists for almost every $x \in \mathbb{R}$ if $f \in L^1(-R,R)$.

Lemma 2.4. *(a)*

$$\left| \ln \left| \frac{H_1(x+iy)}{H_2(x+iy)} \right| - T_y(\xi_1 - \xi_2)(x) \right| \le \pi + 1$$

(b) For almost every $x \in \mathbb{R}$*, we have that*

$$\lim_{y\to 0+} \operatorname{Re} \int_{-R}^{R} \frac{\xi(t)\, dt}{t-x-iy} = (T\xi)(x).$$

Sketch of proof. Part (a) follows from (2.1) by a straightforward calculation, which we leave to the reader; the constant given is not optimal. A similar calculation yields part (b) at all Lebesgue points x of ξ for which, in addition, the limit defining $(T\xi)(x)$ exists. □

It will also be useful to recall the following fact, which was also used in [10].

Lemma 2.5. *Suppose that $\mu_j = c_j = (a_j + b_j)/2$ for all j. Then $\rho_s(K) = 0$.*

Proof. We compare H with the function H_0 that corresponds to $\xi_0 = 1/2$ on $[\min K, \max K]$; in other words, we obtain ξ_0 from ξ by setting $\xi_0 = 1/2$ on each gap. It is easy to confirm that then

$$T_y(\xi_0 - \xi)(x) \geq C \qquad (x \in K, y > 0), \tag{2.7}$$

for some constant C that does not depend on $x \in K$ or $y > 0$. Indeed, $\xi_0 - \xi$ is non-zero only on the gaps of K, and on such a gap (a,b), we have that

$$\xi_0(t) - \xi(t) = \begin{cases} 1/2 & a < t < c \\ -1/2 & c < t < b \end{cases}.$$

Now if also $(a,b) \cap (x-y, x+y) = \emptyset$, then the integral over (a,b) of this function makes a positive contribution to $T_y(\xi_0 - \xi)$, computed according to (2.6): just look separately at the two cases (a,b) to the left of $(x-y, x+y)$ and (a,b) to the right of $(x-y, x+y)$. If the gap intersects $(x-y, x+y)$ but is not contained in this interval, then we can still estimate the contribution to (2.6) by a constant that is independent of $x \in K$ and $y > 0$, and there are at most two such gaps, so (2.7) follows.

By Lemma 2.4(a), this shows that

$$\liminf_{y \to 0+} \left| \frac{H_0(x+iy)}{H(x+iy)} \right| > 0$$

for all $x \in K$. Since $\rho_{0,s} = 0$, Lemma 2.3(b) now implies that ρ has no singular part on K either. □

We are now ready for the

Proof of Theorem 1.2. The claim that δ is upper semicontinuous is just another way of saying that $\mathcal{R}_0(K)$ is compact. Indeed, suppose that $t_n \to t$ is a convergent sequence from $\mathcal{T}$. We can then pick J_n, J_n' from the corresponding (compact!) fibers, that is, $p(J_n) = p(J_n') = t_n$, so that $d(J_n, J_n') = \delta(t_n)$. We can also pass to a subsequence so that

$\delta(t_n)$ approaches the lim sup of the full sequence along this subsequence. Since $\mathcal{R}_0(K)$ is compact, we can further assume here that $J_n \to J$, $J_n' \to J'$. Then $p(J) = p(J') = t$, and thus $\delta(t) \geq d(J, J') = \limsup \delta(t_n)$, as we wished to show.

In particular, it now follows that

$$\{t \in \mathcal{T} : \delta(t) = 0\} = \bigcap_{n \geq 1} \{t : \delta(t) < 1/n\}$$

is a G_δ set, as claimed. To show that this set is also dense, recall from Proposition 2.2 that $\delta(t) = 0$ precisely if $\rho_s(K) = 0$, where ρ is the measure determined by t, identified with $\widehat{\mu}$ here. This happens if, for example, $\mu_j = (a_j + b_j)/2$ is the center of its gap, as in Lemma 2.5, for all $j > j_0$ for some j_0, and $\mu_j \neq a_j, b_j$ for the remaining gaps $j = 1, \ldots, j_0$. Indeed, the H function of such a configuration differs from the one discussed in Lemma 2.5 by a rational factor whose poles and zeros lie in K^c. It now follows from Lemma 2.3(b) that we still have that $\rho_s(K) = 0$ or, equivalently, $\delta(t) = 0$ for any $t \in \mathcal{T}$ corresponding to such a configuration of μ_j's. The set of these t's is clearly dense. $\square$

3. Proof of Theorem 1.3

It will of course suffice to connect an arbitrary $J \in \mathcal{R}_0(K)$ to a specific point J_0, which can be chosen in advance. We will take a J_0 from a trivial fiber. This will greatly simplify the argument because if the associated parameters are denoted by $\widehat{\mu}^{(0)}$, then, as is easy to show and will be verified below, a different set of parameters $(\widehat{\mu}, g(x))$ corresponds to a Jacobi matrix that is close to J_0 precisely if $\widehat{\mu}$ is close to $\widehat{\mu}^{(0)}$ in $\mathcal{T}$. In other words, we may ignore $g(x)$. It is much more difficult to identify neighborhoods of points J with a general set of parameters.

The argument we are going to present would work for an arbitrary J_0 of this type, but it seems reasonable to make a specific choice right away. We will work with $\mu_j^{(0)} = c_j$, as suggested by Lemma 2.5, and $\sigma_j^{(0)} = 1$.

Let $(\widehat{\mu}, g(x))$ be the parameters of a general point $J \in \mathcal{R}_0(K)$. We will approach J_0, starting from this point, by moving the parameters $\widehat{\mu}_j$ to their final destinations $\mu_j = c_j$, $\sigma_j = 1$ step by step, one at a time. We first look at the effect of such a movement in the parameter space.

Lemma 3.1. *Fix arbitrary points $\mu_j \in [a_j, b_j]$ for $j \geq 1$, and consider the parameters (t, μ_j) for $t = \mu_0 \in [a_0, b_0]$. Then the associated measures $\rho_t, \rho_{t,ac}, \rho_{t,s}$ are continuous functions of $t \in [a_0, b_0]$ with respect to the weak-$*$ topology.*

Recall in this context that μ uniquely determines a Krein function ξ and thus also a measure ρ.

Note also that in general, while it is always true that ρ depends continuously on the parameters (μ_j), the individual parts ρ_{ac}, ρ_s need not be continuous; this will only hold in specialized situations such as the one considered in the Lemma.

Proof. It is clear that $\xi_t(x)\,dx$ depends continuously on t, and thus the dependence on t of $H_t(z)$ (with respect to locally uniform convergence on $\mathbb{C}^+$) and ρ_t is also continuous. Hence it suffices to show that $\pi\, d\rho_{t,ac}(x) = |H_t(x)|\chi_K(x)\,dx$ is a continuous function of t.

Let's first look at the situation where $t \to s \in (a_0, b_0)$. Then, from (2.1),

$$\left| \frac{H_t(x)}{H_s(x)} \right| = \left| \frac{s-x}{t-x} \right| ,$$

and this converges to 1 as $t \to s$, uniformly on $x \in K$. Thus clearly $\rho_{t,ac} \to \rho_{s,ac}$ in this case (in fact, in norm).

If $t \to a_0+$, say, and $f \in C_0(\mathbb{R})$ is a test function, then, from the same formula, we obtain that

$$\pi \int f(x)\, d\rho_{t,ac}(x) = \int_K f(x) \left| \frac{a_0 - x}{t-x} \right| |H_{a_0}(x)|\, dx.$$

We know that $H_{a_0} \in L^1(K)$, so dominated convergence now shows that this integral converges to

$$\int_K f(x)|H_{a_0}(x)|\chi_K(x)\,dx = \pi \int f(x)\, d\rho_{a_0,ac}(x)$$

when $t \to a_0+$, as desired. $\square$

Lemma 3.2. *Fix $\widehat{\mu}_j$ for $j \geq 1$, and let $J_t \in \mathcal{R}_0(K)$ be the Jacobi matrix that corresponds to these data, complemented by $\mu_0 = t \in [a_0, b_0]$, $\sigma_0 = 1$, $g_t(x) \equiv 1$. Then J_t is a continuous function of $t \in [a_0, b_0]$.*

Proof. We focus right away on the case of interest here, namely continuity at $s = a_0$ (or $s = b_0$). Continuity at an interior point $s \in (a_0, b_0)$ is easier to establish, with an analogous argument, and will be left to the reader.

So suppose that $t \to a_0$. Since $g_t \equiv 1$, $\sigma_0 = 1$, we have that

$$2\nu_{t,+} = \rho_{t,ac} + 2\rho_{t,s} + \sum_{j\geq 1}(\sigma_j - 1)w_j(t)\delta_{\mu_j}. \tag{3.1}$$

Here, $w_j(t) = \rho_t(\{\mu_j\})$, and, as usual, the sum is over those $j \geq 1$ for which $\mu_j \neq a_j, b_j$; compare (2.4). Similarly,

$$2\nu_{a_0,+} = \rho_{a_0,ac} + 2\rho_{a_0,s} + \sum_{j\geq 1}(\sigma_j - 1)w_j(a_0)\delta_{\mu_j}. \tag{3.2}$$

By Lemma 3.1, $\rho_{t,ac} \to \rho_{a_0,ac}$ and $\rho_{t,s} \to \rho_{a_0,s}$ as $t \to a_0$. So we only need to analyze what happens to the sum in this limit.

It is in fact clear that $w_j(t) \to w_j(a_0)$ for $j \geq 1$. This follows from Lemma 2.3(b), which shows that

$$\frac{w_j(t)}{w_j(a_0)} = \lim_{y\to 0+}\left|\exp\left(\int_{a_0}^{t}\frac{ds}{s-\mu_j-iy}\right)\right| = \exp\left(\int_{a_0}^{t}\frac{ds}{s-\mu_j}\right).$$

Recall in this context that $\mu_j \notin [a_0, b_0]$. In particular, $\mu_j \neq a_0$, and thus this last expression converges to 1 when $t \to a_0$, as asserted.

Next, we claim that if $M \subset (-\infty, a_0)$, then $\rho_t(M) \leq \rho_{a_0}(M)$. This is established by a rather similar argument. We again notice that

$$\left|\frac{H_t(z)}{H_{a_0}(z)}\right| = \left|\frac{z-a_0}{z-t}\right|,$$

observe that this is ≤ 1 for $z = x < a_0$ and refer to Lemma 2.3.

So $\chi_{(-\infty,a_0)}\, d\rho_t = f_t\, d\rho_{a_0}$, and $f_t \leq 1$ here. A similar argument shows that $\chi_{[b_0,\infty)}\, d\rho_t = f_t\, d\rho_{a_0}$, and here $f_t \to 1$ uniformly on $[b_0, \infty)$ as $t \to a_0$.

In particular, given any $\epsilon > 0$, these properties of ρ_t let us find $N \in \mathbb{N}$ and $\delta > 0$ so that

$$\sum_{j>N} w_j(t) < \epsilon \qquad \text{if } a_0 \leq t < a_0 + \delta.$$

This says that the measures $\sum_{j>N} w_j(t)\delta_{\mu_j}$ are uniformly bounded by ϵ in norm for $t \in [a_0, a_0+\delta)$. The norm of a measure controls any metric that generates the weak-$*$ topology. We also saw that $w_j(t) \to w_j(a_0)$ for fixed $j \geq 1$, so it now follows that the measures from (3.1) converge to the one from (3.2). This is what we wanted to show; see Proposition 2.1 and (2.5). $\square$

We are now ready for the

Proof of Theorem 1.3. The overall strategy has already been discussed, at the beginning of this section. We would like to continuously move from $(\widehat{\mu}, g(x))$ to $\widehat{\mu}^{(0)}$, on a parameter interval $0 \leq t \leq 1$, say.

On the interval $0 \le t \le 1/2$, we will move $\widehat{\mu}_1$ to its final destination $\mu_1^{(0)} = c_1$, $\sigma_1 = 1$, and keep the other $\widehat{\mu}_j$'s constant, for the time being.

To make this as challenging as possible, suppose that initially $\sigma_1 = -1$. We then change $g(x)$ to $g(x) \equiv -1$ and keep everything else fixed. In particular, ρ doesn't change, so this clearly can be done along a continuous path in $\mathcal{R}_0(K)$. Now use Lemma 3.2, or rather its analog for $\sigma_0 = -1$, $g \equiv -1$, to move μ_1 to its new value $\mu_1 = a_1$, again keeping everything else fixed. Next, change $g(x)$ to $g(x) \equiv 1$. Recall also that σ_1 plays no role when $\mu_1 = a_1$ is one of the endpoints. Finally, refer to Lemma 3.2, in its original version, one more time to reach a point with $\mu_1 = c_1$, $\sigma_1 = 1$ (and $g(x) \equiv 1$, but this information has become unimportant now). If initially $\sigma_1 = 1$, this point can also be reached, and in fact more easily, by the same procedure with the first two steps omitted.

Next, use the same procedure to move $\widehat{\mu}_2$ to its final value $\widehat{\mu}_2^{(0)}$ on $1/2 \le t \le 3/4$. Continue in this way to define $J(t)$ for all $0 \le t < 1$.

As explained, this function $J(t)$ will be continuous, and obviously $J(0) = J$. To finish the proof of our claim, it remains to show that $\lim_{t\to 1-} J(t) = J_0$. By construction, we clearly have that $\widehat{\mu}(t) \to \widehat{\mu}^{(0)}$ in $\mathcal{T}$. This gives the desired convergence. Indeed, if it was not true that $J(t) \to J_0$ as $t \to 1-$, then, by compactness, $J(t_n) \to J_1 \ne J_0$ along some sequence $t_n \to 1-$. This implies that $\widehat{\mu}(t_n) = p(J(t_n)) \to p(J_1)$, so $p(J_1) = p(J_0)$. However, J_0 was from a trivial fiber, so this information identifies J_1 as $J_1 = J_0$ after all. $\square$

4. Proof of Theorem 1.5

First of all, it is useful to recall the following criterion for the existence of point masses of ρ.

Lemma 4.1. *Let $x \in \mathbb{R}$. Then $\rho(\{x\}) > 0$ if and only if*
$$\int_{x-1}^{x+1} \frac{|\chi_{(x,x+1)}(t) - \xi(t)|}{|t-x|}\, dt < \infty.$$

So, roughly speaking, $\rho(\{x\}) > 0$ if and only if ξ essentially jumps from 0 to 1 at x. For the proof of Lemma 4.1, see, for example, [7, pg. 201] or [9, Lemma 2.4].

In particular, this means that if $x_0 \in K$ is an isolated point of K, then all configurations for which the two adjacent μ_j's are both equal to x_0 will lead to non-trivial fibers. Indeed, we will then have that $\xi = \chi_{(x_0,\infty)}$ on a neighborhood of x_0, so Lemma 4.1 shows that $\rho(\{x_0\}) > 0$.

Let's now begin the

Proof of Theorem 1.5. For convenience, assume that $x_0 = 0$. Denote the two adjacent gaps by $I_1 = (-A, 0)$ and $I_2 = (0, B)$, respectively, and fix $L > 0$ such that $t < L < \min\{A, B\}$.

We will describe the homeomorphism $F : \mathcal{R}_0(K) \to \mathcal{R}_0(K_t)$ that we are required to construct on the level of the spectral data $(\widehat{\mu}, g(x))$. As just explained, if ρ denotes the measure corresponding to a $J \in \mathcal{R}_0(K)$, then $\rho(\{0\}) > 0$ if and only if $\mu_1 = \mu_2 = 0$. In this case, the value of $g_0 \equiv g(0)$ is relevant, and we obtain a non-trivial fiber. We will try to iron out this fiber. We do this, as it were, locally, or perhaps it would be more precise to say that we only rearrange that part of the fiber that corresponds to this parameter $-1 \le g_0 \le 1$. It will thus be useful to slightly reorganize the data of a general $J \in \mathcal{R}_0(K)$, by now listing them as follows:

$$(\widehat{\mu}_j, g(x); \widehat{\mu}_1, \widehat{\mu}_2, g_0)$$

The $\widehat{\mu}_j$, $j \ge 3$, correspond to the remaining gaps, and $g(x)$ is now viewed as a function on $K \setminus \{0\}$. Our map will simply act as the identity on these data $\widehat{\mu}_j$ $(j \ge 3)$, $g(x)$ $(x \in K \setminus \{0\})$, and we will separately map

$$(\widehat{\mu_1}, \widehat{\mu_2}, g_0) \mapsto \left(\widehat{\theta}_1, \widehat{\theta}_2\right), \tag{4.1}$$

with $\theta_1 \in [-A, -t]$, $\theta_2 \in [t, B]$. The new data $(\widehat{\mu}_j, g(x); \widehat{\theta}_1, \widehat{\theta}_2)$ will then correspond to a unique $J' \in \mathcal{R}_0(K_t)$. The induced map $J \mapsto J'$ will be the sought homeomorphism from $\mathcal{R}_0(K)$ onto $\mathcal{R}_0(K_t)$.

To define the map (4.1), we distinguish several cases, which will overlap to some extent:

(i) If $\mu_1 \le -L$, we simply put $\widehat{\theta_1} = \widehat{\mu}_1$. Furthermore, we define $\theta_2 = \mu_2$ if $\mu_2 > L$ and $\theta_2 = t + \mu_2(L - t)/L$ if $0 \le \mu_2 \le L$, and $\widehat{\theta_2} = (\theta_2, \sigma_2)$, where $\sigma_2 = \pm 1$ is the sign of $\widehat{\mu}_2 = (\mu_2, \sigma_2)$.

(ii) The case $\mu_2 \ge L$ is handled similarly: we put $\widehat{\theta_2} = \widehat{\mu}_2$ and $\theta_1 = \mu_1$ or $\theta_1 = -t + \mu_1(L-t)/L$ depending on whether $\mu_1 < -L$ or $-L \le \mu_1 \le 0$, and we again leave the sign σ_1 unchanged. Note that if $-\mu_1 \ge L$ and $\mu_2 \ge L$, so that cases (i) and (ii) both apply, then both definitions lead to the same result.

(iii) The most important case arises when $-\mu_1, \mu_2 \le L$. This region contains the configuration $\mu_1 = \mu_2 = 0$, which leads to the non-trivial fiber. It is now more convenient to define the mapping (4.1) in a more abstract way, rather than give an explicit formula, as we did in the first two cases. We will do this in two steps. We first map the data $(\widehat{\mu}_1, \widehat{\mu}_2, g_0)$, with $-L \le \mu_1 \le 0$, $0 \le \mu_2 \le L$, and $-1 \le g_0 \le 1$ if

$\mu_1 = \mu_2 = 0$, to a plane region that is homeomorphic to a closed disk. We then map the points of this region back to data $(\widehat{\theta}_1, \widehat{\theta}_2)$.

Let us now discuss the first step. The following fact will be used later, and it also helps motivate the procedure that follows. We deal with data associated with the set K here (not K_t).

Lemma 4.2. *Suppose that $\mu_j^{(n)} \to \mu_j$ $(j \ge 3)$,*

$$\mu_1^{(n)} = -(1+G_n)s_n, \quad \mu_2^{(n)} = (1-G_n)s_n,$$

with $-1 \le G_n \le 1$, $G_n \to G$, and $s_n > 0$, $s_n \to 0$. Let ρ_n be the measures of the H functions H_n that correspond to the data $\mu_j^{(n)}$ $(j \ge 1)$, and, similarly, let ρ be the measure that is obtained from their limits μ_j $(j \ge 3)$ and $\mu_1 = \mu_2 = 0$. Then

$$\rho_n(\{\mu_1^{(n)}\}) \to \frac{1}{2}(1+G)\rho(\{0\}).$$

This is a version of the Splitting Lemma from [10]; see Lemma 3.4 of this reference. It gives a precise quantitative description of how the point mass $\rho(\{0\})$ splits into two nearby point masses at $\mu_1^{(n)}$ and $\mu_2^{(n)}$. The statement may look somewhat technical at first sight, but it is really very much at the heart of the matter because it shows how the points of the fiber that are obtained by varying $-1 \le g_0 \le 1$ fit into the space as a whole.

Proof. From (2.1), we obtain the well known product representation

$$H_n(z) = \sqrt{(z-E_-)(z-E_+)}\frac{\sqrt{(z+A)z}}{z-\mu_1^{(n)}}\,\frac{\sqrt{(z-B)z}}{z-\mu_2^{(n)}} \cdot \prod_{j\ge 3} \frac{\sqrt{(z-a_j)(z-b_j)}}{z-\mu_j^{(n)}}. \tag{4.2}$$

Here, $E_- = \min K$, $E_+ = \max K$. We will take absolute values shortly and so do not need to worry about the proper definition of the square roots. Of course, $H(z)$ admits a similar representation, and by comparing the two expressions we obtain that

$$(z-\mu_1^{(n)})H_n(z) = zH(z)P_n(z)\frac{z}{z-(1-G_n)s_n}, \tag{4.3}$$

and here

$$P_n(z) = \prod_{j\ge 3}\frac{z-\mu_j}{z-\mu_j^{(n)}}.$$

We now make use of the formula

$$\rho_n(\{\mu_1^{(n)}\}) = \lim_{y\to 0+} y|H_n(\mu_1^{(n)}+iy)|. \tag{4.4}$$

Observe that $P_n(z) \to 1$ as $n \to \infty$, uniformly in z from a neighborhood of $z = 0$. If we now evaluate the limit from (4.4) with the help of (4.3), we find that

$$\lim_{n\to\infty} \rho_n(\{\mu_1^{(n)}\}) = \frac{1+G}{2} \lim_{z\to 0} |zH(z)|.$$

As $zH(z)$ is holomorphic in a neighborhood of $z = 0$, this latter limit indeed exists, and by the analog of (4.4) for H, it equals $\rho(\{0\})$. □

We are now ready to map our data $(\widehat{\mu}_1, \widehat{\mu}_2, g_0)$, with $0 \le -\mu_1, \mu_2 \le L$, to a plane region, as announced. We again distinguish several overlapping cases:

(a) We map $(0, 0, g_0) \mapsto (0, g_0) \in \mathbb{R}^2$. In the remaining cases, we always assume that $(\mu_1, \mu_2) \neq (0, 0)$.

(b) Next, consider pairs $(\widehat{\mu}_1, \widehat{\mu}_2)$ with $\sigma_1 = 1$, $\sigma_2 = -1$. We map these as follows:

$$(\widehat{\mu}_1, \widehat{\mu}_2) \mapsto (-D, G) \qquad D = \frac{-\mu_1 + \mu_2}{L}, G = \frac{\mu_1 + \mu_2}{\mu_1 - \mu_2} \tag{4.5}$$

Notice how Lemma 4.2 ensures that such a point $(-D, G)$ with $-D$ close to 0 will also correspond to a Jacobi matrix that is close to the one corresponding to the point $(0, G)$. We will discuss this in more detail later. For now, it suffices to observe that the G_n from Lemma 4.2 is indeed related to the $\mu_1^{(n)}$, $\mu_2^{(n)}$ by a formula identical to the one from (4.5).

(c) If $\sigma_1 = -1$, $\sigma_2 = 1$, we use an analogous procedure, but now map to points lying to the right of the segment $\{(0, g) : 0 \le g \le 1\}$. More precisely, we define

$$(\widehat{\mu}_1, \widehat{\mu}_2) \mapsto (D, -G), \tag{4.6}$$

with D, G defined as in (4.5).

(d) If $\sigma_1 = \sigma_2 = 1$, we first map (μ_1, μ_2) to a square which we put on top of the figure obtained so far:

$$(\mu_1, \mu_2) \mapsto \left(\frac{\mu_1}{L}, 1 + \frac{\mu_2}{L} \right).$$

We then fold back the right edge of this square to match the segment $\{(d, 1) : 0 < d \le 1\}$ that we obtained from case (c). To give a more precise description in terms of a formula, write

$$\left(\frac{\mu_1}{L}, 1 + \frac{\mu_2}{L} \right) = (0, 1) + Re_\varphi, \qquad e_\varphi = (\cos\varphi, \sin\varphi),$$

in polar coordinates about $(0,1)$, with $0 \le \varphi \le \pi/2$. The full map (to the plane region) is then given by

$$(\mu_1, \mu_2) \mapsto (0,1) + Re_{2\varphi-\pi}. \tag{4.7}$$

(e) Finally, if $\sigma_1 = \sigma_2 = -1$, attach similarly a piece to the bottom of the region constructed so far.

Recall that the parameter σ_j is not used if $\mu_j = 0$ $(j = 1, 2)$. The reader should check that this does not cause consistency problems in the above definitions.

This map is clearly bijective onto its image, and this image is obviously homeomorphic to a closed disk. Moreover, the preimage of its boundary is given by the set of all $(\widehat{\mu}_1, \widehat{\mu}_2, g_0)$ for which $\mu_1 = -L$ (and $\sigma_1 = 1$ or $= -1$) or $\mu_2 = L$.

We can therefore map this image homeomorphically onto

$$\{(\widehat{\theta}_1, \widehat{\theta}_2) : t \le -\theta_1, \theta_2 \le L\},$$

where we now view this set as the Cartesian product of two circular arcs, endowed with the product topology. Moreover, by the observation on the preimage of the boundary that was just made, we may further insist that the resulting composite map

$$(\widehat{\mu}_1, \widehat{\mu}_2, g_0) \mapsto (\widehat{\theta}_1, \widehat{\theta}_2) \tag{4.8}$$

agrees with the ones defined earlier, in cases (i), (ii) above, on $\mu_1 = -L$ and on $\mu_2 = L$. In fact, we now see that we can glue together all three maps, with no consistency problems. This completes our definition of a function that maps as indicated in (4.1).

As announced earlier, we now build a map on the complete parameter spaces by simply leaving the other parameters unchanged:

$$(\widehat{\mu}_j, g(x); \widehat{\mu}_1, \widehat{\mu}_2, g_0) \mapsto (\widehat{\mu}_j, g(x); \widehat{\theta}_1, \widehat{\theta}_2) \tag{4.9}$$

Recall again that the $\widehat{\mu}_j$ $(j \ge 3)$ correspond to the other gaps, and $g(x)$ is a Borel function on $x \in K \setminus \{0\}$, $-1 \le g(x) \le 1$.

This map accomplishes the desired task of straightening out the fiber associated with $x_0 = 0$. To prove this claim, we have to verify several properties, and we do this in separate steps.

Step 1: There is a well defined map $F : \mathcal{R}_0(K) \to \mathcal{R}_0(K_t)$ that is induced by (4.9). Clearly, a set of parameters as displayed on the right-hand side of (4.9) uniquely determines a $J \in \mathcal{R}_0(K_t)$. However, recall that $g(x)$ is not uniquely determined by $J \in \mathcal{R}_0(K)$; rather, any function $h(x)$ with $h = g$ almost everywhere with respect to ρ_s on $K \setminus \{0\}$ will represent the same J. We must show that such a change of representative of g does not affect the Jacobi matrix represented by the right-hand side of (4.9).

This in fact follows immediately from the observation that the two ρ measures involved here are mutually absolutely continuous with respect to each other on $\mathbb{R} \setminus (-A, B)$. Indeed, if we again use product representations of the type (4.2) and compare ξ functions, then we see that the two H functions that are associated with (4.9) differ from each other by a factor of the form

$$\frac{(z-\mu_1)(z-\mu_2)}{(z-\theta_1)(z-\theta_2)} \frac{\sqrt{(z-t)(z+t)}}{z}. \tag{4.10}$$

If also $\mu_1 \neq -A$, $\mu_2 \neq B$, then this expression approaches non-zero limits for $z = x + iy$, $y \to 0+$, $x \in \mathbb{R} \setminus (-A, B)$. Thus our claim about the equivalence of the ρ measures follows in the usual way from Lemma 2.3.

The case where $\mu_1 = -A$ or $\mu_2 = B$ (or both) is treated similarly. No extra work is needed, really; if, say, $\mu_1 = -A$, then, by construction, $\theta_1 = -A$ also, and (4.10) in properly reduced form is still valid.

Step 2: The map $F : \mathcal{R}_0(K) \to \mathcal{R}_0(K_t)$ induced by (4.9) *is injective.* To verify this, we first observe that two different triples $(\widehat{\mu}_1, \widehat{\mu}_2, g_0)$ never get mapped to the same point $(\widehat{\theta}_1, \widehat{\theta}_2)$ under (4.8). This is clear from the construction. So injectivity could fail only if two data sets that only differ in their $g(x)$ function got mapped to the same Jacobi matrix. However, the equivalence of the ρ measures that was discussed in the previous step prevents this from happening.

Step 3: F is surjective onto $\mathcal{R}_0(K_t)$. This is again obvious from the construction: any set of parameters $\widehat{\mu}_j$, $-1 \le g(x) \le 1$, $-A \le \theta_1 \le -t$, $t \le \theta_2 \le B$, $\sigma_1 = \pm 1$, $\sigma_2 = \pm 1$ is obtained as an image under (4.9), and these exhaust the whole parameter space.

Step 4: F is continuous. Let $J_n, J \in \mathcal{R}_0(K)$, and assume that $J_n \to J$. We must show that then also $F(J_n) \to F(J)$.

From Proposition 2.1 we know that the corresponding parameters satisfy $\widehat{\mu}_j^{(n)} \to \widehat{\mu}_j$ $(j \ge 1)$, and $\nu_{n,+} \to \nu_+$ in weak-$*$ sense.

The main issue will be to show that, similarly,

$$(\widehat{\theta}_1^{(n)}, \widehat{\theta}_2^{(n)}) \to (\widehat{\theta}_1, \widehat{\theta}_2). \tag{4.11}$$

This is done by considering in detail how the map (4.8) was defined. It is quite clear, of course, that we will obtain this convergence if $(\widehat{\mu}_1, \widehat{\mu}_2, g_0)$ lies in the interior of one of the regions defined by cases (i), (ii), or (iii) (a)–(e) above. We will therefore focus right away on those points where several cases overlap. In fact, we will explicitly discuss only two representative scenarios and leave the remaining cases to the reader.

In the second scenario below, we will be confronted with the crucial question of how the fiber fits into the space as a whole, but as a warm-up, let us first look at a situation where $\mu_1 = 0$, $0 < \mu_2 < L$, $\sigma_2 = 1$. Note that for large n, we will then have that also $0 < \mu_2^{(n)} < L$ and $\sigma_2^{(n)} = 1$. Moreover, $\mu_1^{(n)} \to 0$, but both $\sigma_1^{(n)} = -1$ and $\sigma_1^{(n)} = 1$ are possible for these approximating points, and we are either in case (c) (if $\sigma_1^{(n)} = -1$) or in case (d) (if $\sigma_1^{(n)} = 1$) from above. In the first case, we notice that $G_n \to -1$, so the image points under the intermediate map (4.6) will converge to $(\mu_2/L, 1)$. In the second case, we have to follow the recipe that led to (4.7). Now clearly $(\mu_1^{(n)}/L, 1 + \mu_2^{(n)}/L) \to (0, 1 + \mu_2/L)$, hence $R_n \to \mu_2/L$, $\varphi_n \to \pi/2$, and thus the points from the right-hand side of (4.7) will again approach $(\mu_2/L, 1)$. Similar considerations are in fact also needed to confirm that the overlapping cases give consistent definitions. In any event, this point $(\mu_2/L, 1)$ is obtained as the image of $\mu_1 = 0$, $0 < \mu_2 < L$, $\sigma_2 = 1$.

As (4.8) was defined as the composition of this map with a continuous map, we have indeed verified (4.11) in this case.

The most interesting scenario arises when $\mu_1 = \mu_2 = 0$; at such points, cases (a), (b), and (c) meet. In fact, additional issues arise here when either $g_0 = 1$ or $g_0 = -1$ because then cases (d) and (e), respectively, are also involved. These are handled by the procedure discussed in the preceding paragraph, so we will give in to temptation here and restrict our discussion to the easier case $-1 < g_0 < 1$.

We must again verify that the intermediate image point $(0, g_0)$ from case (a) is also approached if the approximating points lie in the regions defined by cases (b) or (c). Let's discuss case (b) explicitly, the other case being analogous, of course. So consider points $(\mu_1^{(n)}, \mu_2^{(n)}) \neq (0,0)$, $\mu_j^{(n)} \to 0$ $(j = 1, 2)$, $\sigma_1^{(n)} = 1$, $\sigma_2^{(n)} = -1$. Clearly, $D_n \to 0$, and Lemma 4.2 indeed makes sure that $G_n \to g_0$. Let us elaborate on this in more detail: if we didn't have this convergence, say $G_n \to g_1 \neq g_0$ on a subsequence, then Lemma 4.2 would show that

$$\chi_{(-d,d)}\nu_{n,+} \to \frac{1}{2}(1 + g_1)\rho(\{0\})\delta_0$$

on a suitable small neighborhood $(-d, d)$ of 0. Here we use the fact that the measures $\nu_{n,+}$, restricted to such a set, are point measures with a single point mass at $\mu_1^{(n)}$, of weight $\rho_n(\{\mu_1^{(n)}\})$. This we read off from (2.4), recalling that $\sigma_1^{(n)} = 1$, $\sigma_2^{(n)} = -1$. Also, Lemma 4.1 makes sure that $\rho_n(\{0\}) = 0$.

However, $\chi_{(-d,d)}\nu_+ = (1/2)(1+g_0)\rho(\{0\})\delta_0$, so we have reached a contradiction to Proposition 2.1. Thus $G_n \to g_0$, and this again implies that (4.11) holds. This concludes our discussion of this claim.

To finally show that $F(J_n) \to F(J)$, consider an arbitrary limit point $J_0 \in \mathcal{R}_0(K_t)$ of this sequence. We use the same notation as above. The J_n, J have data as on the left-hand side of (4.9); an additional index n is used to label the data of J_n. We must then show that J_0 can only be the unique Jacobi matrix from $\mathcal{R}_0(K_t)$ with data $\widehat{\mu}_j$ ($j \geq 3$), $g(x)$, $\widehat{\theta}_1$, $\widehat{\theta}_2$. Since $\mathcal{R}_0(K_t)$ is compact, this will yield the desired convergence.

From Proposition 2.1 and what has just been discussed, we in fact already know that J_0 must have data $\widehat{\mu}_j$ ($j \geq 3$) and $\widehat{\theta}_1$, $\widehat{\theta}_2$.

To obtain information on the function $g(x)$ that is associated with J_0, we take a look at the ν_+ measures of the $F(J_n)$, $F(J)$, that is, the ν_+ measures that are determined by the data from the right-hand side of (4.9). To distinguish these from the measures of the original Jacobi matrices J_n, J, we denote them by $\widetilde{\nu}_{n,+}$, $\widetilde{\nu}_+$.

The argument is similar to what we did in step 1. Suppose that $\mu_1 \neq -A$, $\mu_2 \neq B$, and let I be an interval of the form $I \equiv (-A+\epsilon, B-\epsilon)$ that contains μ_1 and μ_2. Again, by (4.10),

$$\chi_{\mathbb{R}\setminus I}\, d\widetilde{\nu}_{n,+}(x) = \chi_{\mathbb{R}\setminus I} \left| \frac{(x-\mu_1^{(n)})(x-\mu_2^{(n)})}{(x-\theta_1^{(n)})(x-\theta_2^{(n)})} \frac{\sqrt{x^2-t^2}}{x} \right| d\nu_{n,+}(x), \tag{4.12}$$

and of course such a formula also holds for $\widetilde{\nu}_+$ and ν_+. As $n \to \infty$, the factors on the right hand side converge to 1 uniformly on $x \in I^c$. Hence the fact that $\nu_{n,+} \to \nu_+$ also implies that

$$\chi_{I^c}\widetilde{\nu}_{n,+} \to \chi_{I^c}\widetilde{\nu}_+. \tag{4.13}$$

This conclusion is also valid if $\mu_1 = -A$ or $\mu_2 = B$. The observations made in step 1 in this context are still relevant: if, say, $\mu_1 = -A$, then $\mu_1^{(n)} = \theta_1^{(n)}$ for all large n and the corresponding factors from (4.12) simply cancel each other out.

Now $\widetilde{\rho}_s(K \cap I) = 0$, so (4.13) indeed forces any limit point of the sequence $\widetilde{\nu}_{n,+}$ to employ the same function $g(x)$, almost everywhere with respect to $\chi_K\widetilde{\rho}_s$, as the measure $\widetilde{\nu}_+$, if represented as in (2.4). □

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