Math 5863
Archive of True/False Problems
(with brief answers)

1. January 18 (from Topology I Final)

1.1. *A space X which is compact, Hausdorff and second countable is metrizable.*

TRUE: Compact Hausdorff spaces are normal and the Urysohn Metrization Theorem applies.

1.2. *Every separable first countable space is second countable.*

FALSE: $\mathbb{R}_\ell$ is one counterexample.

1.3. *With the cofinite topology $\mathbb{R}$ is path connected.*

TRUE: If Y has the cofinite topology then a function $f : X \to Y$ is continuous iff each point-inverse $f^{-1}(y)$ is closed in X. In particular if X is T_1 and f is one-to-one (or even finite-to-one) then f is continuous. (The interval $I = [0, 1]$ is T_1.)

1.4. *If (X, d) is a metric space, $x \in X$ and $\epsilon > 0$ then the closure of $B(x, \epsilon)$ equals $\{y \in X \mid d(x, y) \leq \epsilon\}$.*

FALSE: Consider the discrete metric on the two point set $X = \{a, b\}$ with the discrete topology.

1.5. *Let $\mathcal{T}$ be the topology on $\mathbb{R}$ generated by the basis $\mathcal{B} = \{(a, \infty) \mid a \in \mathbb{R}\}$. A continuous function from $(\mathbb{R}, \mathcal{T})$ to $\mathbb{R}_\ell$ must be constant.*

TRUE: Consider an open set in $U \subseteq \mathbb{R}_\ell$ and a continuous function $f : \mathbb{R}_\mathcal{T} \to \mathbb{R}_\ell$. We know that since f is continuous that $f^{-1}(U) \subseteq \mathbb{R}_\mathcal{T}$ must be open and, thus, of the form (a, ∞) for some $a \in \mathbb{R}$. However, let $V \subseteq U \in \mathbb{R}_\ell$ be open. We can see that $(b, \infty) = f^{-1}(V) \subseteq f^{-1}(U) = (a, \infty)$ with $b \leq a$. Thus, we must have that f is constant on $f^{-1}(U) \cap f^{-1}(V)$ for all U and V open in $\mathbb{R}_\ell$. Therefore, we must have that f is a constant function.

1.6. *Let X be a space and $A \subseteq X$ and let $\mathcal{B}$ be a sub-basis for the topology on X. If every $B \in \mathcal{B}$ contains an element of A then A is dense in X.*

FALSE:

2. January 25

2.1. *The topology on $\mathbb{R}$ with basis $\mathcal{B} = \{(a, \infty) \mid a \in \mathbb{R}\}$ is completely metrizable.*

FALSE: The space generated by this basis is not Hausdorff. Therefore, it is not metrizable.

2.2. *The Cantor one-third set in $\mathbb{R}$ is complete.*

TRUE: Pick your favorite point in $\mathcal{C}$. You can go down far enough in the construction that you can find more points in $\mathcal{C}$ that it is within any epsilon of your choice.

2.3. *The French railway metric on $\mathbb{R}^2$ is complete.*

TRUE: Where is the only place that a Cauchy Sequence can converge?

2.4. *Total boundedness is a topological property of metric spaces.*

FALSE: Notice that $(0, 1)$ is homeomorphic to $\mathbb{R}$. We know that $(0, 1) \subseteq [0, 1]$ is totally bounded, but $\mathbb{R}$ is not totally bounded.

2.5. *A separable metric space is totally bounded.*

FALSE: $\mathbb{R}$ is second countable (so separable), but $\mathbb{R}$ is not totally bounded.

2.6. *There is a metric on the open interval $X = (0, 1)$ generating the Euclidean topology for which $X^* - X$ is uncountable (where X^* denotes the completion of the metric space X).*

TRUE: For any isometric spaces X and Y, the sets$X^* \setminus X$ and $Y^* \setminus Y$ have the same cardinalities. In our case, $X = (0, 1)$ and $Y = \{(x, sin(1/x)) \mid 0 < x < 1\}$. Note that $Y^* \setminus Y = \{(0, y) \mid y \in [-1, 1]\}$. We pull the usual metric on Y into X via the map f from X to Y defined by $f(x) = (x, sin(1/x))$ such that f is an isometry together with this new metric on X.

3. February 6

3.1. *The French railway metric on $\mathbb{R}^2$ is locally compact.*

FALSE: Consider the unit circle in $\mathbb{R}^2$.

3.2. *Let X be the space obtained by identifying together each pair of opposite edges of a (filled-in) regular octagon in $\mathbb{R}^2$ (using the same orientation on the opposite edges). Then X is homeomorphic to the torus T^2.*

FALSE: Consider the Euler Characteristic.

3.3. *Contractible spaces are connected.*

TRUE: Continuous image of a connected set is connected.

3.4. *The Euler characteristic of the Mobius band M^2 is 0.*

True: This should be pretty straightforward. There is 1 two-cell, 3 one-cells, and 2 zero-cells.

3.5. *There is a "capital letter subspace"[1] of R^2 that has Euler characteristic 2.*

FALSE: We can draw these out and calculate the Euler Characteristic.

3.6. *Contractible spaces are locally connected.*

FALSE: Think of a the comb space or a space with an infinite number of rays originating from a single point. We can take a neighborhood at a point on the y-axis that does not contain the x-axis and see that this space is not locally connected.

3.7. *Every continuous function from a space X to a contractible space Y is nullhomotopic. (A map is nullhomotopic iff it is homotopic to a constant function.)*

TRUE: Let $f : X \to Y$ be such a continuous function and $i : Y \to Y$ be the identity map that is homotopic to a constant map. Then, we can consider the composition $i \circ f : X \to Y$ which is a homotopic to a constant map. Therefore, we can see that $i \circ f$ is nullhomotopic. Moreover, notice that the choice of continuous function was arbitrary.

4. February 20

4.1. *For evey topological space X the empty subset is a deformation retract of X.*

FALSE: The empty set is not an element of X, so I cannot map anything to the empty set.

4.2. *The singleton set consisting of the origin is a deformation retract of R^2 with the topology induced by the French railway metric.*

TRUE: We can think of the retraction that continuously maps each point of $\mathbb{R}^2$ to the origin by moving the point along the line that it forms from itself to the origin. This created a deformation retract from $\mathbb{R}^2$ to the origin.

4.3. *Let X and Y be topological spaces with $y_0 \in Y$. The (projection) map $p_1(x, y) = (x, y_0)$ from $X \times Y$ to $X \times \{y_0\}$ is a retraction.*

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[1] ala Chapter 0 of Hatcher's book

TRUE: Notice that for any set of the form $U \times \{y_0\}$ we have $p_1(u, y) = (u, y_0)$ for any $(u, y_0) \in U \times \{y_0\}$. Therefore, we have that p_1 is a retraction of $X \times Y$ to $X \times \{y_0\}$.

4.4. *In the previous problem if $Y = I^2$ then p_1 is a deformation retraction.*

TRUE: Notice that the retraction $p_1(x, y)$ is isotopic to the identity function on X. Therefore, we see that p_1 is a deformation retraction.

4.5. *If $f : I \to X$ is a path from x_0 to x_0 in X then f is null-homotopic.*

FALSE: Let $f : I \to X$ be a loop from x_0 to x_0 in the space X. Define $H : I \times I \to X$ by $H(s,t) = f((1-t)s + t)$. This is a composition $H = f \circ K$ of the linear function $K : I \times I \to I$ defined by $K(s,t) = (1-t)s + t$ with f, and is continuous because both K and f are continuous. Observe that $H(s, 0) = f(s)$ and $H(s, 1) = f(1) = x_0 = c_{x_0}(s)$. Therefore H is a homotopy from f to the constant function $c_{x_0} : I \to X$. By definition, we conclude that f is null-homotopic.
(The important thing to notice here is that the homotopy H is definitely not a homotopy relative to the endpoints $0, 1$ of I. But that's OK because we just need to show that f and the constant map c_{x_0} are homotopic, and not that they are path-homotopic.)

4.6. *A surface with odd Euler characteristic is non-orientable.*

TRUE: We know that the Euler Characteristic of a non orientable surface is $2 - g$ where g is the genus of the surface. However, for an orientable surface, we have the Euler Characteristic being $2(1 - g)$ which is always even.

4.7. *The surfaces with id patterns $aba^{-1}b$ and c^2d^2 are homeomorphic.*

TRUE: Draw out $aba^{-1}b$ do some cutting and pasting.

4.8. *The 2-disk with identification id $abcac^{-1}ab$ is a nonorientable surface with genus* 3.

FALSE: Notice that the letter a appears three times. Therefore, we know that this is not a surface because a surface must have pairs of letters or one letter that appears once (for a surface with boundary).

4.9. *The 2-disk with id pattern $abcac^{-1}ab$ has Euler characteristic* -1.

TRUE: We have 1-zero cell, 3-one cells, and 1-two cell! Thus, $\chi(X) = 1 - 3 + 1 = -1$.

4.10. *There is a 2-disk with id pattern X representing the torus T^2 for which radial projection gives a deformation retraction from $X - \{origin\}$ onto a subspace of X homeomorphic to $\{(x, 0) \mid -2 \le x \le 2\} \cup \{(0, y) \mid 1 \le y \le 2\} \cup \{(x, y) \mid x^2 + y^2 = 1\}$.*

TRUE: We can draw the cell complex and see that there are 6 zero cells, and 7 one cells. Therefore, when we calculate the Euler Characteristic, we see that $\chi(X) = 6 - 7 = -1$. Thus, the Euler Characteristic of the torus and X, there is a homeomorphism between them.

4.11. *If X is a contractible space then $\{x_0\}$ is a deformation retract of X for some $x_0 \in X$.*

False: