# Topology II Final Exam
# May 7, 2019

Instructions: Work seven problems, at least three from Part I and three from Part II.

PART I:

1. (a) Define what it means for two continuous functions to be homotopic.
(b) Define what it means for two topological spaces to be homotopy equivalent.
(c) If two path connected spaces have isomorphic fundamental groups must they be homotopy equivalent? Give a brief justification for your answer.

2. Let $f$ and $g$ be loops in $X$ based at $x_0$, and let $\overline{f}$ and $\overline{g}$ be their reverses. Show that $f$ and $g$ are path homotopic if and only if $\overline{f}$ and $\overline{g}$ are.

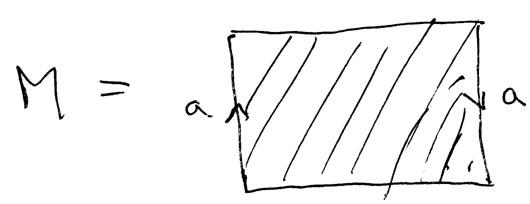
3. Let $f : (X, x_0) \to (Y, y_0)$ be a continuous function.
(a) Define the homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and carefully explain why it is well-defined.
(b) Show that $f_*$ need not be surjective if $f$ is surjective.

4. (a) Give definitions of what it means for a topological space (i) to be simply connected, and (ii) to be contractible.
(b) Compare and contrast the two properties in (a). Does either one imply the other? Give examples as necessary to support your comparison.

5. Let $X$ be a path connected space with open and simply connected subsets $U$ and $V$ with $X = U \cup V$.
(a) Show that if $U \cap V$ is path connected then $\pi_1(X)$ is trivial.
(b) Show that $\pi_1(X)$ may be non-trivial if $U \cap V$ is not path connected.

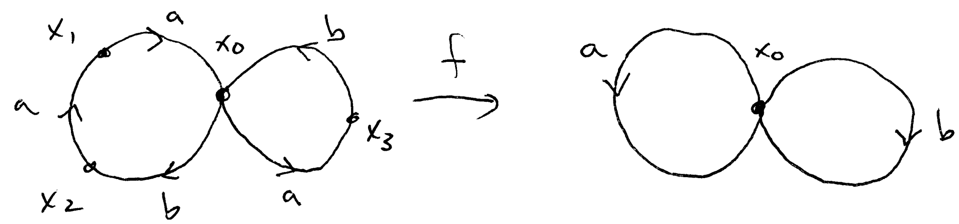
PART II:

6. Let $M$ be the closed Mobius band as pictured below.
(a) Describe a 1-dimensional cell complex that is homotopy equivalent to $M$.
(b) Describe the universal cover of $M$.
(c) Describe all possible covering maps $p : \widetilde{X} \to M$ up to isomorphism.

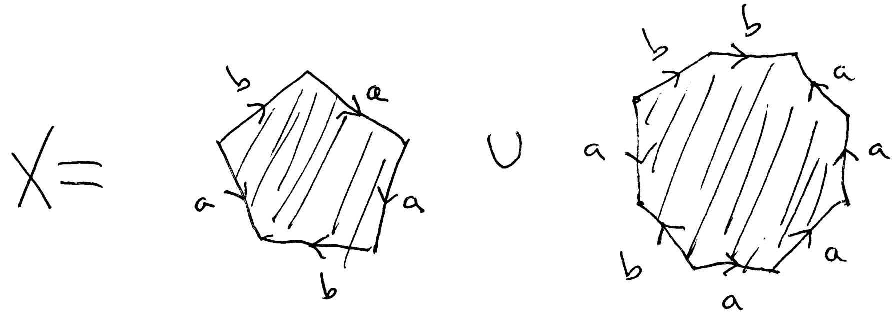



7. Let $X = S^1 \vee S^1$ be the wedge of two circles.
(a) Give examples of two different 6-sheeted covering spaces of $X$ with one being regular and the other not.
(b) Describe the group of covering transformations of your two examples from part (a).
(c) Is there a 6-sheeted covering of $X$ whose group of covering transformations is isomorphic to the dihedral group with 6 elements? Explain.

8. Let $X = S^1 \vee S^1$ and let $f : X \to X$ be the continuous surjection function shown in the picture below (where each $x_i$ gets sent to $x_0$ and each open labeled segment in the domain gets sent homeomorphically to its counterpart in the codomain with the same label). Let $H$ be the subgroup of $\pi_1(X, x_0) \cong F(a, b)$ given by $H = f_*(\pi_1(X, x_0))$.
(a) Is $f$ a covering map? Explain briefly.
(b) Give a set of generators for the subgroup $H$.
(c) Draw a picture of the covering space of $X$ associated with the subgroup $H$.
(d) Construct a 4-sheeted covering map $p : \widetilde{X} \to X$ for which $f : X \to X$ lifts (that is, there is a continuous function $\widetilde{f} : X \to \widetilde{X}$ with $p\widetilde{f} = f$).




9. Let $X$ be the 2-dimensional cell complex pictured below.
(a) How many cells of each dimension does $X$ have?
(b) Give a presentation for the fundamental group of $X$.
(c) Show that $X$ is not simply connected.




10. Use fundamental groups to show that $S^1 \times S^1$ is not a retract of $S^1 \times D^2$.