Compact Coverings

Problems #3 on page 79 of Hatcher's book and #6 in section 53 of Munkres book ask for a proof of a fundamentally important covering space result:

If $p : \widetilde{X} \to X$ is a finite-sheeted covering map whose base space X is compact then the total space $\widetilde{X}$ is compact.

Although not difficult, the proof is somewhat nuanced. Here is a brief outline.

Proof. Assume that X is compact and $p : \widetilde{X} \to X$ is a k-sheeted covering map. Let $\mathcal{U}$ be an open covering of $\widetilde{X}$. It must be shown that $\mathcal{U}$ has a finite subcover.

Let $x \in X$ and choose sets $U_1(x), \ldots, U_k(x)$ in $\widetilde{\mathcal{U}}$ such that

$$p^{-1}(x) \subset U_1(x) \cup \cdots \cup U_k(x).$$

Since covering maps are open maps, each of the sets $p(U_i(x))$ is open in X. Let V_x be a neighborhood of x which is contained in

$$p(U_1(x)) \cap \cdots \cap p(U_n(x)).$$

The collection $\{V_x \mid x \in X\}$ is an open cover of X, and it has a finite subcover since X is compact. The subcover will equal $\{V_{x_1}, \ldots, V_{x_n}\}$ for some elements $x_1, \ldots x_n \in X$. Then it is not hard to check that $\{U_i(x_j) \mid 1 \leq i \leq k, 1 \leq j \leq n\}$ is a finite subcover of $\mathcal{U}$, as desired. □

NOTE: With slight modifications this argument actually shows that if $f : X \to Y$ is a continuous, open map onto a compact space Y and each point-inverse $p^{-1}(y)$ is finite then X is compact.