The Fibonacci Numbers $(a_n)_{n=0}^{\infty}$ are defined inductively by the 2-step recursion

$$\begin{cases} a_n = a_{n-1} + a_{n-2} & \text{for } n \geq 2 \quad (1) \\ a_0 = 0 \text{ and } a_1 = 1 \end{cases}$$

The first few terms are:

$$\{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \underset{a_{10}}{55}, 89, \dots\}.$$

To understand these numbers in a different way we look for solutions to (1). To this end define

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

These two real numbers are the solutions to the quadratic equation

$$x^2 - x - 1 = 0$$

(as can be checked using the quadratic formula). Observe that this means that

$$\alpha + 1 = \alpha^2 \quad \text{and} \quad \beta + 1 = \beta^2. \quad (2)$$

<u>Claim 1</u> Putting $a_n = \alpha^n$ gives a solution to the recursion equation (1), however it does not give a formula for the Fibonacci numbers because it satisfies : $a_0 = \alpha^0 = 1$ and $a_1 = \alpha^1 = \frac{1+\sqrt{5}}{2}$.

<u>Verification for Claim 1</u>: Suppose $a_n = \alpha^n$. Then

$$a_{n-1} + a_{n-2} = \alpha^{n-1} + \alpha^{n-2} = \alpha^{n-2}(\alpha+1) = \alpha^{n-2} \cdot \alpha^2 = \alpha^n = a_n$$

using (2), and this verifies the recursion (1). □

<u>Claim 2</u> Putting $a_n = \beta^n$ gives another solution to the recursion (1), which is different from Claim 1.

<u>Verification for Claim 2</u> It's just like for Claim 1 because $\beta + 1 = \beta^2$. □

<u>Claim 3</u> If C and D are any real numbers then putting

$$a_n = C\alpha^n + D\beta^n \quad (3)$$

is another solution to the recursion (1).

<u>proof sketch</u> Assume that $a_n = C\alpha^n + D\beta^n$. Then

$$\begin{aligned} a_{n-1} + a_n &= (C\alpha^{n-1} + D\beta^{n-1}) + (C\alpha^{n-2} + D\beta^{n-2}) \\ &= C(\alpha^{n-1} + \alpha^{n-2}) + D(\beta^{n-1} + \beta^{n-2}) \\ &= C\alpha^n + D\alpha^n = a_n \end{aligned}$$

using Claims 1 and 2. □

For the solution (3) observe that

$$\begin{cases} a_0 = C + D, & \text{and} \\ a_1 = C\alpha + D\beta \end{cases}$$

Thus if we choose $C = \frac{1}{\sqrt{5}}$ and $D = -\frac{1}{\sqrt{5}}$ then

$$\begin{cases} a_0 = \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{5}} = 0 \\ a_1 = \frac{1}{\sqrt{5}}\alpha - \frac{1}{\sqrt{5}}\beta = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) = 1. \end{cases}$$

This shows:

<u>Theorem</u> For each $n \in \mathbb{Z}_{\geq 0}$, the n^{th} Fibonacci number is $a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$. (4)

<u>some comments</u>

(i) $\beta = \frac{1-\sqrt{5}}{2} \approx -.61803$. So $\beta^n = \left(\frac{1-\sqrt{5}}{2}\right)^n \to 0$ as $n \to \infty$. On the other hand, $\alpha = \frac{1+\sqrt{5}}{2} \approx 1.61803$. So $\alpha^n \to \infty$ as $n \to \infty$. This shows that for large integers n the n^{th} Fibonacci number is very close to $\frac{1}{\sqrt{5}}\alpha^n$. It follows that the Fibonacci numbers exhibit exponential growth.

(ii) Since β^n is so close to 0 we can actually describe the nth Fibonacci number as being the integer which is closest in value to $\frac{1}{\sqrt{5}}\alpha^n$.

(iii) Note that it is rather surprising that $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$ is an integer for every $n \geq 0$. (This is guaranteed by the Theorem.)

(iv) The formula (4) is theoretically interesting but may not be terribly useful for calculating Fibonacci numbers. For example try calculating $a_{10} = 55$ by expanding $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{10} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{10}$ by hand.