Example: $W_n = \{\text{bit strings of length } n\}$, $n \in \mathbb{N}$

$F: W_n \to W_n$ is defined by

$F(w)$ = the bitstring obtained from w by replacing each 0 with a 1 and each 1 with a 0.

where $w \in W_n$. Observe that

$$F(F(w)) = w, \quad \text{for each } w \in W_n \qquad (1)$$

Then F has an inverse function and $F^{-1} = F$.

Show that F is a bijection:

① F is one-to-one.

Suppose that w_1 and w_2 are bit strings in W_n which satisfy $F(w_1) = F(w_2)$. Then, using (1),

$$w_1 = F(F(w_1)) = F(F(w_2)) = w_2.$$

This shows that F is injective.

② F is onto. = codomain(F)

Suppose that $w \in W_n$. Let $x = F(w)$. $\in W_n$ Then

$$F(x) = F(F(w)) = w.$$

This shows that every element $w \in W_n$ equals $F(x)$ for some $x \in W_n$. Therefore F is onto.

Problem How many bit strings of length 19 have an odd number of 0's?

Consider W_{19}, for which we know that $|W_{19}| = 2^{19}$.

Define subsets of W_{19}: ← Problem Find $|A|$.

$$A = \{w \in W_{19} \mid w \text{ has an odd number of } 0\text{'s}\}$$

$$B = \{w \in W_{19} \mid w \text{ has an even number of } 0\text{'s}\}$$

Then $A \cup B = W_{19}$ and $A \cap B = \emptyset$, so the sum principle implies that

$$|A| + |B| = |W_{19}| = 2^{19}. \qquad (2)$$

Next observe that the function $F: W_{19} \to W_{19}$ (which was defined above) satisfies that

$$F(A) = B$$

because if $w \in W_{19}$ has k 0's where k is odd then $F(w)$ has k 1's, which in turn implies that the number of 0's in $F(w)$ is $19-k$ which is even.

(*) Since $F: W_n \to W_n$ is a bijection and $F(A) = B$ the "restriction function"

$F|_A : A \to B$ is a bijection. Therefore, using (2),

by Relabelling Principle.

$$2^{19} = |A| + |B| = 2|A| \qquad \text{b/c } |A| = |B|$$

which means that

$$|A| = 2^{18} = 262{,}144 \qquad 2|A| = 2^{19} \Rightarrow |A| = \frac{2^{19}}{2} = 2^{18}$$

Problem How many bit strings of length 20 have an odd number of 0's?

First note that if A is the set of bit strings in W_{20} with an odd number of 0's and B is the set with an even number of 0's. Then

$$F(A) = A \quad \text{and} \quad F(B) = B \qquad F(A) \neq B$$

So the previous argument doesn't work.

Instead consider subsets of A: ← Problem $|A| = ??$

$$A_1 = \{w \in A \mid \text{first bit in } w \text{ is } 1\}$$

$$A_2 = \{w \in A \mid \text{first bit in } w \text{ is } 0\}$$

Then A is the disjoint union of A_1 and A_2 and

$$|A| = |A_1| + |A_2| \qquad (3)$$

To count $|A_1|$ notice that an element of $w \in A_1$ is comprised of 1 followed by a bit string of length 19 with an odd number of 0's.

By the previous problem $|A_1| = 2^{18}$.

Similary an element of A_2 consists of a 0 followed by a bit string of length 19 with an even number of 0's. By the previous problem $|A_2| = 2^{18}$. From (3) we get

$$|A| = 2^{18} + 2^{18} = 2^{19}$$