Example Show that $\binom{2n}{2} = n^2 + 2\binom{n}{2}$

(Hammack: Problem 5, page 122)

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Let $X$ be a set with $2n$ elements,

Then $\binom{2n}{2}$ is the number of 2 element subsets of $X$. Let's write $X = X_1 \cup X_2$ where $X_1$ and $X_2$ are subsets of $X$ with $n$ elements each, and disjoint.

To count the 2-element subsets of $X$ will break into two parts

✓ • Both elements are in one of $X_1$ or $X_2$

✓ • One element is in $X_1$ and the other is in $X_2$

If both elements are in $X_1$, then there are $\binom{n}{2}$ of these. Similarly if both elements are in $X_2$ then $\binom{n}{2}$ of these.

If one element is in $X_1$ and the other is in $X_2$ then there are $\binom{n}{1}\binom{n}{1}$ of these subsets. so

$$\binom{2n}{2} = \text{2 element subsets of } X = \binom{n}{2} + \binom{n}{2} + \binom{n}{1}\binom{n}{1}$$

$$= 2\binom{n}{2} + n^2 \qquad \text{since } \binom{n}{1} = n. \quad \square$$

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Or prove this algebraically:

$$\binom{2n}{2} = \frac{(2n)!}{2!\,(2n-2)!} = \frac{(2n)(2n-1)}{2} = n(2n-1)$$

$$2\binom{n}{2} + n^2 = 2\,\frac{n(n-1)}{2} + n^2 = n^2 - n + n^2$$