Math 2513
Some review problems for Exam 2

Problem 1. (a) Describe what it means for a function $f : A \to B$ not to be onto.
(b) Show that the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = |n + 10| - 10$ is not onto.
(c) Explicitly describe $range(f)$ for the function f of part (b) and then prove your assertion.

Problem 2. Prove: If $f : A \to B$ and $g : B \to C$ are onto functions then $g \circ f$ is onto.
(note: $g \circ f$ is the "composition" of g and f defined by $g \circ f(a) = g(f(a))$.)

Problem 3. A **bit string of length** n is a sequence of n 0's and 1's. Let A be the set consisting of all bit strings of length 5. Let $f : A \to \mathbb{Z}$ be the function which assigns to each bit string the number of 1's which it contains. Let $g : A \to \mathbb{Z}$ be the function defined by

$$g(x_1x_2x_3x_4x_5) = 2^0x_1 + 2^1x_2 + 2^2x_3 + 2^3x_4 + 2^4x_5$$

where $x_1x_2x_3x_4x_5$ is a bit string of length 5.
(a) What is the range of f?
(b) What is the range of g?
(c) Is the function f injective? Justify your answer.
(d) Is the function g injective?

Problem 4. (a) How many bit strings of length 12 don't include a 01 substring?
(b) How many bit strings of length 12 don't contain a 11 substring?

Problem 5. Let x be a real number. Use mathematical induction to show that $x^0 + x^1 + x^2 + \cdots + x^n = (1 - x^{n+1})/(1 - x)$ for each integer $n \geq 0$.

Problem 6. Show that congruence modulo 17 is an equivalence relation on $\mathbb{Z}$.

Problem 7. (a) How many bit strings of length 12 contain exactly 10 ones?
(b) How many bit strings of length 12 contain exactly 10 ones but do not contain two consecutive zeros?
(c) How many bit strings of length 12 contain at least 10 ones?
(d) How many bit strings of length 12 contain at least 10 ones but do not contain eleven consecutive ones?

Problem 8. Consider words formed using the letters $\{a, b, c, d, e, f, g, h\}$ which have no repeated letters.
(a) How many of these words have length 10?
(b) How many of these words have length 6?
(c) How many of these words have length 6 and don't include the letter g?
(d) How many of these words have length 6 and do include the letter g?
(e) How many of these words have length 3 or smaller? (Don't forget the empty word which has length 0.)

Problem 9. Prove that $2\binom{k}{2} + k^2 = \binom{2k}{2}$, for each $n \in \mathbb{N}$.

Problem 10. Let m be an integer.
(a) Prove that if $m^2 + m + 1 = 0 \pmod{65}$ then $m^2 + m + 1 = 0 \pmod{13}$.
(b) Express the converse of the statement in (a) and give an example showing that it is false.

Problem 11. How many bijections from $\mathbb{Z}_{15}$ to itself are there for which each element gets sent to another element that has the same remainder when divided by 5?

Problem 12. If k and n are natural numbers with $k \leq n$ then how many injective functions are there from a set with k elements to a set with n elements?

Problem 13. How many bit strings of length six have each of the following properties?
(a) start with 010 and end with 111
(b) start with 010 or end with 111
(c) contain five successive 1's
(d) do not contain five successive $1's$
(e) contain a substring with the form 0001
(f) do not contain a substring with the form 0001

Problem 14. A coin is tossed ten times in succession and a sequence of H's (heads) and T's (tails) is obtained. How many ways are there to do this so that there are exactly seven H's in the sequence?

Problem 15. Let m be a natural number. Consider solutions for x of the equations $4 \times x = 1$ and $4x + 7 = 0$ in $\mathbb{Z}_m$.
(a) When $m = 17$ what are the solutions to each of the equations?
(b) Do either of the equations have solutions when $m = 20$?

Problem 16. This problem involves counting sets of positive integers which have four digits. (So this consists of all integers n with $1000 \leq n \leq 9999$.)
(a) Use the product principle to determine the total number of these integers.
(b) How many four digit integers have four distinct digits?
(c) How many four digit integers are divisible by 3?
(d) How many four digit integers are there in which all either all digits are even of all digits are odd?
(e) How many four digit integers read the same forwards and backwards? (These are called "palindromic integers".)
(f) How many four digit integers contain exactly one 3?
(g) How many four digit integers contain exactly one 3 and exactly one 4?

ANSWERS: (a) $9,000$, (b) $4,536$, (c) $3,000$, (d) $1,125$, (e) 90, (f) 2025 , (g) 720