Discrete Mathematics
More counting problems—with brief answers

1. How many 5-letter words using only A's, B's, C's and D's are there that do not contain the word BAD?
   answer: $976 = 4^5 - 3 \cdot 4^2$

2. How many 10-letter words using only A's, B's, C's and D's are there that either start or end with BAD are there?
   answer: $32,512 = 4^7 + 4^7 - 4^4$

3. How many 10-letter words using only A's, B's, C's and D's are there which have 3 A's, 2 B's, 3 C's and 2 D's but do not contain the word AB?
   answer:
   $$11,760 = \binom{10}{3,2,3,2} - \left(9\binom{8}{2,1,3,2} - \binom{8}{2,1,3,2}\right)$$
   COMMENTS: It is convenient to use the 'multinomial' notation in which
   $$\binom{10}{3,2,3,2} = \binom{10}{3}\binom{7}{2}\binom{5}{3}\binom{2}{2} = \frac{10!}{3!2!3!2!}$$
   This number represents the total number of words, including all those which have an AB subword. From this, subtract off the number of words that have an AB subword. (In counting these, observe that there can't be more than two AB subwords because only two B's can be used. The result is obtained by adding together the number of words that have AB starting in position i, where i is between 1 and 9, then subtracting off the number of words that have two AB subwords.)

4. How many bit strings of length twelve don't include a '01' substring?
   answer: 13

5. How many bit strings of length 12 don't contain a '11' substring?
   answer: 377

6. How many nonnegative integer solutions are there to the equation
   $$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 32?$$
   answer: $435,897 = C(37,5)$

7. How many positive integer solutions are there to the equation
   $$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 32?$$
   answer: $169,911 = C(31,5)$

8. How many positive integer solutions are there to the inequality
   $$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 < 32?$$
   (Hint: consider $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 32$.)
   answer: $736,281 = C(31,6)$

9. How many nonnegative integer solutions are there to the inequality
   $$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 < 32?$$
   answer: $2,324,784 = C(37,6)$

10. How many arrangements of the letters of RECURRENCERELATION have no two vowels adjacent?
answer: $10,478,160,000 = \binom{10}{4,2,2,1,1} C(11,8) \binom{8}{1,4,1,1,1}$
RECALL the "multinomial" notation:

$$\binom{10}{4,2,2,1,1} = C(10,4) \times C(6,2) \times C(4,2) \times C(2,1) \times C(1,1) = \binom{10}{4}\binom{6}{2}\binom{4}{2}\binom{2}{1}\binom{1}{1}.$$

11. How many arrangements of the letters of RECURRENCERELATION have the vowels in alphabetical order?
answer: $91,891,800 = C(18,8)\binom{10}{4,2,2,1,1}$.
First choose 8 of the 18 positions in which to place the vowels A,E,E,E,E,I,O,U (there is only one way to arrange them in alphabetic order). Then position the remaining letters in the ten remaining slots.

12. How many ways can 8 persons, including Peter and Paul, sit in a row with Peter and Paul not sitting next to each other?
answer: $30,240 = P(8,8) - 2 \cdot P(7,7) = 8! - 2 \times 7!$

13. How many ways can 8 persons, including Peter and Paul, sit at a round table with Peter and Paul sitting next to each other?
answer: $1,440 = 2P(6,6)$

14. How many ways can 4 persons of each of n nationalities stand in a row with each person standing next to a fellow national?
answer: $(2n)!(4!)^n/2^n$

15. How many ways are there to distribute 30 green balls to 4 persons if Alice and Eve together get no more than 20 and Lucky gets at least 7?
answer: 2464

16. How many ways are there to select a dozen doughnuts chosen from 7 varieties with the restriction that at least 1 doughnut of each variety is chosen?
answer: $462 = C(11,6)$. (Consider 5 stars and 6 bars)

17. How many ways are there to assign 50 agents to 5 different countries so that each country gets 10 agents?
answer: $48334775757901219912115629238400 = \binom{50}{10,10,10,10,10}$

18. How many ways are there to put 17 red balls into 12 distinguishable boxes with at least 1 ball in each box?
answer: $4368 = C(16,11) = C(16,5)$

19. How many ways can 9 dice fall (unordered)?
answer: $2002 = C(14,5)$

20. How many ways are there to arrange 5 C's and 15 R's such that there are at least 2 R's between any 2 C's?
answer: $792 = C(12,5)$

21. How many ways are there to select 5 integers from $\{1,2,\ldots,20\}$ such that the (positive) difference between any two of the five is at least 3?
answer: $792 = C(12,5)$
COMMENT: Think of arrangements with 5 stars (representing the five integers to be chosen) and 19 bars (representing 20 different boxes in which each integer is to be placed). The condition that the difference between two integers is at least 3 means that we are only interested in arrangements that have at least 3 bars between any two successive stars. So start by putting 3 bars between 5 stars, this leaves 7 bars to be placed in 5 possible positions.

22. How many possible outcomes (unordered) are there if k dice are tossed?
answer: $C(k+5,k)$

23. How many bit strings of length 5 are there that either start with 000 or end with 111?
answer: 8

24. How many bit strings of length n where $n > 5$ are there that either start with 000 or end with 111?
answer: $15 \cdot 2^{n-6} = 2^{n-3} + 2^{n-3} - 2^{n-6} = 15 \cdot 2^{n-6}$

25. How many 4-letter words are there with the letters in alphabetical order?
answer: $23,751 = C(29,25)$.
Note that once we choose the letters to be used there is only one way to arrange them in alphabetical order.

26. How many 4-letter words are there with no letter repeated and the letters in alphabetical order?
answer: $14,950 = C(26,4)$

27. How many ways can we partition 18 persons into study groups of 5, 6 and 7?
answer: $14,702,688 = \binom{18}{5,6,7}$

28. How many ways can we partition 18 persons into study groups of 5, 5 and 4?
answer: 0

29. How many ways can we partition 18 persons into 3 study groups of 6?
answer: $2,858,856 = \binom{18}{6,6,6}/3!$

30. How many arrangements of 7 R's and 11 B's are there such that no two R's are adjacent?
answer: $792 = C(12,7)$
First put the B's down. Then choose 7 out of the 12 interstices in which to put the R's.

31. How many ways are there to give each of 5 children 4 of 20 distinguishable toys?
answer: $305,540,235,000 = \binom{20}{4,4,4,4,4}$

32. How many ways can 10 men and 7 women sit in a row so that no two women are next to each other?
answer: $6,035,420,160,000 = 10!C(11,7)7! = P(10,10)P(11,7)$

33. How many ways can 10 men and 7 women sit at a round table so that no two women are next to each other?
answer: $219,469,824,000 = P(9,9)P(10,7)$

34. How many 3-letter words are there with no repeated letter if the middle letter is a vowel?
answer: $3,000 = 5 \cdot P(25,2)$ (This is not counting 'y' as a vowel.)

35. How many 5-card poker hands are there with two pairs?
answer: $123,552 = C(44,1)C(13,2)C(4,2)^2$

36. How many arrangements of the letters in MISSISSIPPI have at least two adjacent I's?
answer: $27,300 = \binom{11}{4,4,2,1} - \binom{7}{4,2,1}\binom{8}{4}$

37. How many arrangements of the letters in MISSISSIPPI have no two I's adjacent?
answer: $7,350 = \binom{7}{4,2,1}\binom{8}{4}$

38. How many arrangements of the letters in MISSISSIPPI have no P adjacent to an S? (Hint: Although it is the same problem, it is easier to consider no S adjacent to a P.)
answer: $4725 = C(5,4)C(6,2)C(7,4) + \binom{6}{4,1,1}C(8,4)$.
Consider two cases (i) where the two P's are adjacent and (ii) where they are not adjacent. In case (i), there are $\binom{6}{4,1,1}$ ways to put down one M, four I's and one PP–once these are put down then we must choose how to distribute four S's in 5 interstices (two of the interstices are ruled out for being adjacent to the PP) and there are $C(8,4)$ ways to do this (4 stars and 4 bars). In case (ii) there are $C(5,4)$ ways to arrange one M and four I's, now choose two different interstices in which to put two P's (there are $C(6,2)$ ways to do this) and finally we must choose how to distribute four S's in 4 interstices (four of the interstices are ruled out for being adjacent to a P) and there are $C(7,4)$ ways to do this (4 stars and 3 bars).

39. How many possible outcomes are there if a pair of dodecahedral dice, with sides numbered 1 through 12, are thrown?
answer: $78 = C(13,2)$

40. How many different selections of fruit can be made from 5 oranges and 7 apples?
answer: $6 \cdot 8 = 48$

41. How many different words of at least one letter can be made from 3 A's and 3 B's?
answer: $68 = 2 + 2^2 + 2^3 + (2^4 - 2) + (2^5 - 2 - 10) + C(6, 3)$

42. How many ways can we partition mn distinguishable objects into m piles of n objects each?
answer: $\binom{mn}{n, n, \cdots, n}/m!$