Discrete Math Group Project #9
10/23/20 (due by 6PM on Wednesday, 10/28)

PART I:

Problem #1. (a) In the document *Writing Proofs* by Christopher Heil (available at the course web site), locate at least ten sentences which contain an exclamation point (!) or some words in all caps, and write out each of these sentences.
(b) Give three bullet point take-aways from these sentences listing things that will be important to focus on in your own writing. (Write in English sentences please!)

PART II: The integers mod m
Throughout this part m will denote a fixed positive integer. The "set of integers modulo m" is the set with m elements $\{[0],[1],[2],[3],\dots,[m-2],[m-1]\}$, which we will denote by $\mathbb{Z}_m$. These elements can be added, subtracted and multiplied according to the rules

$$[a]+[b]=[r] \text{ where } r \text{ is the remainder when the sum } a+b \text{ is divided by } m$$

$$[a]-[b]=[s] \text{ where } s \text{ is the remainder when the difference } a-b \text{ is divided by } m$$

and

$$[a]\cdot[b]=[s] \text{ where } s \text{ is the remainder when the product } ab \text{ is divided by } m.$$

In other words, these rules show that $\mathbb{Z}_m$ has a system of arithmetic on it for each $m \in \mathbb{N}$. In these systems most of the standard laws of arithmetic hold. This includes the commutative laws for (addition and multiplication), the associative laws, the distributive law, additive and multiplicative identities, and additive inverses. (However multiplicative inverses do not always exist.)
In order to simplify notation, in writing the elements of $\mathbb{Z}_m$ it is common to eliminate the square brackets and simply write $\mathbb{Z}_m=\{0,1,\dots m-1\}$. With this notation, the additive identity is 0 (because $a+0=0=0+a$), the multiplicative identity is 1 (because $1\cdot a=a=a\cdot 1$), and the additive inverse of a is $m-a$ (because $a+(m-a)=m=0$). If $a\in\mathbb{Z}_m$ has a multiplicative inverse b then $a\cdot b=1$.

Problem #2. The set $\mathbb{Z}_6$ has six elements $0,1,2,3,4,5$. Explain that in this system $3+4=1$, $3\times 4=0$, $3^2=3$, and $2^{100}=4$.

Problem #3. Calculate the value in $\mathbb{Z}_{15}$ of each of: $10+3$, $10+7$, $(6\cdot 2+2\cdot 3)+3$, $3\cdot(4\cdot 2)$, $(3\cdot 4)\cdot 2$, $2(2+2\cdot 3)+3$, $(7+1)^2$, $5\cdot 4-4\cdot 6$, and 2^{16}.

Problem #4. Make a table which lists the additive inverse for each of the ten elements of $\mathbb{Z}_{10}$, and also lists the multiplicative inverse for each element when it exists.

Problem #5. Show that each non-zero element of $\mathbb{Z}_{11}$ has a multiplicative inverse and make a table listing these.

Problem #6. Consider the function $f:\mathbb{Z}_m\to\mathbb{Z}_m$ defined by $f(a)=a^2$. Show that this function is not one-to-one. If an element is in the range of this function f then we say that it has a square root.

Problem #7. Which elements of $\mathbb{Z}_7$ have square roots? Which elements of $\mathbb{Z}_{15}$ have square roots? Make tables listing the values of the square roots of each element that has one.

Problem #8. The quadratic equation $3x^2+5x+3=0$ does not have any real solutions, does it have any solutions in Z_{15}? If so, find them.

Problem #9. When does a quadratic equation $ax^2+bx+c=0$ have a solution in $\mathbb{Z}_6$ (where $a,b,c\in\mathbb{Z}_6$)? Explain.
(possible hint: what happens if you try to apply the quadratic formula?)