Discrete Math Group Project #11
(due by 6PM on Wednesday, 11/11)

PART I: An interesting bijection.

PROBLEM #1. Consider the function $G : \mathbb{N} \to \mathbb{Z}$ defined by the rule

$$G(n) = \frac{(-1)^n(2n-1)+1}{4}.$$

(a) Determine the value of $G(n)$ when n is even. Determine the value of $G(n)$ when n is odd. Use your answers to give a piecewise description of the function G with the form

$$G(n) \;=\; \begin{cases} --- & \text{if } n \text{ is even} \\ --- & \text{if } n \text{ is odd} \end{cases}$$

[Important Note: Your answer to this problem should make it clear that $G(n)$ is an integer for each natural number n, and that G is a function from $\mathbb{N}$ to $\mathbb{Z}$ as stated. (From the original definition of G it is only clear immediately that $G(n)$ will be a rational number of the form $\frac{p}{4}$ where p is an integer.)]
(b) Show that G is an injective function.
(Suggestion: Let m and n be integers with $m \neq n$, show that $G(m) \neq G(n)$ by considering three separate cases: (case i): one of m and n is odd and the other is even: (case ii): both m and n are even; (case iii): both m and n are odd.)
(c) Show that $G : \mathbb{N} \to \mathbb{Z}$ is a bijection.
(d) List the first ten terms of the sequence $(G(n))_{n=1}^{\infty}$. Does this give a better understanding of why G is a one-to-one correspondence?

PART II: Counting functions.

PROBLEM #2. Let $X = \{a, b, c, d\}$ and let $Y = \{1, 2\}$.
(a) How many functions $f : X \to Y$ are there? List them.
(b) How many functions $g : Y \to X$ are there? List them.
(c) How many of the functions $f : X \to Y$ are surjective? List them.
(d) How many of the functions $f : X \to Y$ are injective? List them.
(e) How many of the functions $f : Y \to X$ are surjective? List them.
(f) How many of the functions $f : Y \to X$ are injective? List them.

PROBLEM #3. Let A be a set with four elements.
(a) How many bijections are there from A to itself?
(b) How many injections are there from A to itself?
(c) How many surjections are there from A to itself?
(d) If $A = \{a_1, a_2, a_3, a_4\}$ then how many bijections $f : A \to A$ satisfy that $f(\{a_1, a_2\}) = \{a_1, a_2\}$?[1]
(e) How many bijections are there from the power set $\mathcal{P}(A)$ to itself?

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[1]Here we are using the definition that if $f : A \to B$ is a function and $C \subseteq A$ then $f(C) = \{f(c) \mid c \in C\}$, which is a subset of B. In this way we see that each function $f : A \to B$ determines a function from $f : \mathcal{P}(A) \to \mathcal{P}(B)$. (At first it may be confusing, but it is standard convention to use the same letter f to denote the function between the power sets.)