Class Problem
Math 2513
Thursday, June 16

PROBLEM. Let $f : A \to B$ and $g : B \to C$ be functions. Consider the implication $\mathcal{P}$:

If $f : A \to B$ and $g : B \to C$ are one–to–one functions then $g \circ f : A \to C$ is a one–to–one function.

(a) State the converse of $\mathcal{P}$.
(b) State the contrapositive of $\mathcal{P}$.
(c) State the inverse of $\mathcal{P}$.
(d) Two of the statements among $\mathcal{P}$, its converse, its contrapositive and its inverse are false. Determine which two are the false ones and explain.

SOLUTION:

(a) *If $g \circ f : A \to C$ is one–to–one then both $f : A \to B$ and $g : B \to C$ are one–to–one.*

(b) *If $g \circ f : A \to C$ is not one–to–one then $f : A \to B$ and $g : B \to C$ are not both one–to–one.* This can also be phrased as: *If $g \circ f : A \to C$ is not one–to–one then $f : A \to B$ is not one–to–one or $g : B \to C$ is not one–to–one.*

(c) *If $f : A \to B$ and $g : B \to C$ are not both one–to–one then $g \circ f : A \to C$ is not one–to–one.* This can also be phrased as: *If $f : A \to B$ is not one–to–one or $g : B \to C$ is not one–to–one then $g \circ f : A \to C$ is not one–to–one.*

(d) Both $\mathcal{P}$ and its contrapositive are true, but the converse and the inverse of $\mathcal{P}$ are false.
To see that the converse is false: let $f : \{1\} \to \{1, 2\}$ be given by $f(1) = 1$ and let $g : \{1, 2\} \to \{1\}$ be given by $g(1) = g(2) = 1$. Then $g \circ f : \{1\} \to \{1\}$ is given by $g \circ f(1) = 1$, and $g \circ f$ is a one–to–one function. However g is not one–to–one, which shows that the converse of $\mathcal{P}$ is false.
Since the inverse of $\mathcal{P}$ is logically equivalent to the converse of $\mathcal{P}$, the same example shows that the inverse is false.