Class Problem
Math 2513
Monday, June 13

Problem. Consider the two functions f and g which are defined by:

$f : \mathbb{R} \to (0, \infty)$ where $f(x) = \frac{1}{x^2+1}$ for each real number x, and
$g : (0, \infty) \to (0, \infty)$ where $g(t) = \frac{1}{t^2+1}$ for all positive real numbers t.

(1) Explain carefully why f and g are different functions.
(2) Determine the range of f.
(3) Show that the function f is not one–to–one.
(4) Show that the function g is one–to–one.

Notes: (a) Write in sentences please. (b) In the definition of the functions, $(0, \infty)$ represents the open interval from 0 to $+\infty$ in the real line; this coincides with the set of positive real numbers.

(1) The domain of f is the set $\text{domain}(f) = \mathbb{R}$ of real numbers while the domain of g is the set $\text{domain}(g) = (0, \infty)$ of positive real numbers. Since these two sets are not equal, the two functions are different. (The domain is part of the definition of a function, as is the codomain also.)

(2) The range of f is $f(\mathbb{R}) = (0, 1] = \{t \in \mathbb{R} \mid 0 < t \leq 1\}$. (As a real-valued function, f is an even function, it has an absolute maximum at $x = 0$, it is decreasing on the interval $(0, \infty)$ and $\lim_{x\to+\infty} f(x) = 0$. This example is a good reminder of how difficult it can be to determine the range of a function.)

(3) As an example, $\sqrt{5}$ and $-\sqrt{5}$ are two distinct elements in the domain of f (they are both real numbers). Since $f(\sqrt{5})$ and $f(-\sqrt{5})$ both equal $1/26$, this shows that f is not a one–to–one function.

(4) Let s and t be elements of $(0, \infty)$ for which $g(s)$ equals $g(t)$. Then $\frac{1}{1+s^2} = \frac{1}{1+t^2}$. Therefore $1 + s^2 = 1 + t^2$ which implies that $t = \pm s$. Since both s and t are positive, t cannot equal $-s$ (since s is positive, $-s$ must be negative). Therefore t must equal s. We have shown that if $s, t \in (0, \infty)$ and $g(s) = g(t)$ then $s = t$. By the definition of one–to–one it follows that g is one–to–one.