EXAM 2
Math 2513
10-15-04

1. *(15 points)* Prove that there are infinitely many prime integers.
ANSWER: See page 156 of the textbook.

2. *(20 points)* Use the Principle of Mathematical Induction to prove that: $2(1 + 3 + 3^2 + \cdots + 3^n) = 3^{n+1} - 1$ for every positive integer n.
ANSWER: Let $\mathcal{P}(n)$ be the statement that "$2(1 + 3 + 3^2 + \cdots + 3^n)$ equals $3^{n+1} - 1$". Proceed by verifying the Basis Step and the Inductive Step.

3. *(10 points)* State the result that you are asked to prove in problem 2 using summation notation.
ANSWER: $2\sum_{i=0}^{n} 3^i = 3^{n+1} - 1$

4. *(20 points)* Determine whether each of the following is true or false and then prove your assertion.
   (a) If a, b and c are positive integers such that c divides $a - b$ then c divides a.
   (b) If a, b and c are positive integers such that c divides a and c divides b then c divides $a - b$.
   (c) There are no integers a and b such that $a^2 + 4b^2 = 26$.
ANSWER: (a) FALSE, (b) TRUE, (c) TRUE.

5. *(10 points)* Let A and B be sets such that $\overline{A} \subseteq B$. Show that $\overline{B} - A = \emptyset$.
ANSWER: Let A and B be sets with $\overline{A} \subseteq B$. Suppose that $\overline{B} - A$ is nonempty. Then there is an element x in $\overline{B} - A$. By the definition of set difference this means that $x \in \overline{B}$ and $x \notin A$. By the definition of complement, $x \in \overline{A}$. Since $\overline{A} \subseteq B$ this implies that $x \in B$. So we have $x \in \overline{B}$ and $x \in B$. It follows that $x \notin B$ and $x \in B$, which is a contradiction. Therefore the supposition that $\overline{B} - A$ is nonempty must be false, so $\overline{B} - A = \emptyset$. $\square$

6. *(15 points)* Illustrate the Euclidean algorithm by using it to find the greatest common divisor of 42 and 140.
ANSWER: $140 = 3 \cdot 42 + 14$ and $42 = 3 \cdot 14 + 0$, so the greatest common divisor is 14.

7. *(10 points)* (a) Find at least two positive integers n which satisfy $n \equiv 7 \pmod{11}$.
   (b) Find at least two negative integers n which satisfy $n \equiv 4 \pmod{23}$.
   (c) Find at least two integers n which satisfy both $n \equiv 7 \pmod{11}$ and $n \equiv 4 \pmod{23}$.
ANSWER: (a) Any n with the form $n = 11k + 7$ where $k \in \mathbb{N}$ will work.
(b) Any n with the form $n = -23k + 4$ where $k \in \mathbb{N}$ will work.
(c) Any n with the form $n = 253k + 73$ where $k \in \mathbb{Z}$ will work.