EXAM 3 – Brief Answers
Math 2513
4-14-05

1. *(15 points)* (a) Carefully state the principle of inclusion–exclusion.
(b) Illustrate part (a) by counting the number of bit strings of length 8 which contain exactly 5 ones or which start with "0110".

ANSWER:
(a) Let A be a finite set with $A = A_1 \cup A_2$ for two subsets A_1 and A_2 of A. Then

$$|A| = |A_1| + |A_2| - |A_1 \cup A_2| \ .$$

(b) Let A be the set of bit strings of length 8 which contain exactly 5 ones or which start with "0110". Let A_1 be the set of bit strings of length 8 which contain exactly 5 ones, and let A_2 be the set of bit strings of length 8 which start with "0110". Then $A = A_1 \cup A_2$. Then $|A_1| = C(8, 5) = 56$ and $|A_2| = 2^4 = 16$. The intersection $A_1 \cap A_2$ consists of bit strings of length 8 which start with "0110" and have exactly five ones, and this set has $C(4, 3) = 4$ elements. Then, by the principle of inclusion/exclusion,

$$|A| = |A_1| + |A_2| - |A_1 \cap A_2| = 56 + 16 - 4 = 68.$$

2. *(20 points)* Use the technique of mathematical induction to prove that $\sum_{k=2}^{n} 2k + 1 \ = \ (n + 3)(n - 1)$ for every integer $n \geq 2$.

ANSWER:
For each integer $n \geq 2$ let $\mathcal{P}(n)$ be the proposition

$$\mathcal{P}(n) : 5 + 7 + 9 + \cdots + (2n + 1) = (n + 3)(n - 1)$$

Basis Step: The LHS of the proposition $\mathcal{P}(2)$ is 5 while its RHS is $(2 + 3)(2 - 1) = 5$. Since these values are equal this shows that $\mathcal{P}(2)$ is true.
Inductive Step: Assume that $\mathcal{P}(k)$ is true for some $k \geq 2$. Thus $5 + 7 + 9 + \cdots + (2n + 1) = (k + 3)(k - 1)$. The LHS of $\mathcal{P}(k + 1)$ is $5 + 7 + 9 + \cdots + (2(k + 1) + 1)$, and we have

$$5+7+9+\cdots+(2(k+1)+1) = (5 + 7 + 9 + \cdots + (2k + 1))+(2k+3) = (k+3)(k-1)+(2k+3) = k^2+4k = (k+4)k.$$

Since the RHS of $\mathcal{P}(k + 1)$ is $((k + 1) + 3)((k + 1) - 1) = (k + 4)k$, this shows that $\mathcal{P}(k + 1)$ is true.
Therefore we conclude that $\mathcal{P}(n)$ is true for all $n \geq 2$ using the principle of mathematical induction.

3. *(10 points)* Write out the entire row of Pascal's triangle whose first two entries are 1 and 9.

ANSWER:
The tenth row of Pascal's triangle consists of the ten numbers $\binom{9}{k}$ where $0 \leq k \leq 9$. The numbers are

$$\{1, 9, 36, 84, 126, 126, 84, 36, 9, 1\}.$$

4. *(15 points)* Let S be the set of all words of length 10 that can be formed using the letters A, B and C where repetition of letters is allowed.
(a) How many elements does S have?
(b) How many subsets with exactly 3 elements does S have? Show that there more than a million.
(c) Let S_1 be the subset of S consisting of words where at most two of the three letters appear. Determine $|S_1|$.
(d) How many elements of S contain an "$AABBCCAA$" subword?
(e) How many elements of S contain exactly three A's, five B's and two C's?

ANSWER:
(a) $|S| = 3^{10}$.
(b) $C(3^{10}, 3) = 3^{10}(3^{10}-1)(3^{10}-2)/6$. This number can easily be shown to be larger than $3^{24} = (27)^8 > 10^8$.
(c) $|S_1| = 3 \cdot 2^{10} - 3 = 3069$.
(d) 27
(e) 2520

5. *(10 points)* (a) How many bit strings contain exactly 4 zeros and 6 ones if every 0 is immediately followed by a 1?
(b) List the bit strings in part (a).

ANSWER:
(a) $C(6, 2) = 15$

6. *(10 points)* Prove the identity $C(2n, 2) = 2C(n, 2) + n^2$ where n is a positive integer.

ANSWER:
Let n be a positive integer. If $n \geq 2$ then $C(n, 2) = n!/(2!(n-2)!)$ and we have

$$C(2n, 2) = \frac{(2n)!}{2!(2n-2)!} = \frac{(2n)(2n-1)}{2} = 2n^2 - n$$

and

$$2C(n, 2) + n^2 = 2\frac{n!}{2!(n-2)!} + n^2 = n(n-1) + n^2 = 2n^2 - n.$$

Since the resulting are equal the identity is proved for $n \geq 2$. When $n = 1$ then $C(n, 2) = 0$ and both sides of the equation evaluate to 1.

7. *(10 points)* (a) Determine the coefficient of x^9y^3 in the expansion of $(x + y)^{12}$
(b) Determine the coefficient of x^9y^3 in the expansion of $(x - 3y)^{12}$
(c) Determine the coefficient of x^0 in the expansion of $(x + \frac{1}{x^3})^{12}$

ANSWER:
(a) 220 (b) -5940 (c) 220

8. *(10 points)* How many solutions to the equation $n_1 + n_2 + n_3 = 20$ are there if
(a) n_1, n_2 and n_3 are positive integers?
(b) n_1, n_2 and n_3 are integers larger than 5?

ANSWER:
(a) 171 (b) 6