THE PICARD–VESSIOT ANTIDERIVATIVE CLOSURE

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Abstract. F is a differential field of characteristic zero with algebraically closed field of constants C. A Picard–Vessiot antiderivative closure of F is a differential field extension $E \supset F$ which is a union of Picard–Vessiot extensions of F, each obtained by iterated adjunction of antiderivatives, and such that every such Picard–Vessiot extension of F has an isomorphic copy in E. The group G of differential automorphisms of E over F is shown to be prounipotent. When C is the complex numbers and $F = C(t)$ the rational functions in one variable, G is shown to be free prounipotent.

Introduction

Throughout this work, F denotes a differential field of characteristic zero with derivation $D = D_F$ and algebraically closed field of constants C.

For terminology and basic results regarding differential Galois extensions of F with linear algebraic differential Galois group (Picard–Vessiot extensions) we refer to [M]. If $E \supset F$ is a differential field extension, we will denote the derivation D_E by D when no ambiguity arises. If S is a subset of E, we let $F\langle S\rangle$ denote the smallest differential subfield of E that contains both F and S. When S is itself a differential subfield of E, $F\langle K\rangle = K\langle F\rangle$ (which also equals $K\langle F\rangle$) is called the compositum of F and K. We denote the field of constants of E by C_E. The extension has *no new constants* if $C_E = C$. For an element y of any extension E, we use y' and $y^{(n)}$ to denote $D(y)$ and $D^n(y)$ as usual. We always use $G(E/F)$ to denote the group of differential automorphisms of E over F.

Definition. An element y of a differential extension $E \supset F$ will be called an *antiderivative* if $y' \in F$. We will call $E \supset F$ an *antiderivative extension* if there are elements $y_1, \dots, y_n$ in E which differentially generate E over F and such that each y_i' belongs to the differential field generated over F by $y_1, \dots, y_{i-1}$. If $n = 1$, the antiderivative extension will be called *simple*.

The following facts are easy to see, and will be shown below: an antiderivative extension is generated as a field, and not just as a differential field, by the elements $y_1, \dots, y_n$; a proper simple antiderivative extension without new constants is a Picard–Vessiot extension with differential Galois group the additive group $\mathbb{G}_a$;

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and therefore any antiderivative extension without new constants is a tower of $\mathbb{G}_a$ Picard–Vessiot extensions.

A tower of Picard–Vessiot extensions need not itself be Picard–Vessiot, or even embedable in a Picard–Vessiot extension. (This is well known; we include an antiderivative extension example below for $F = \mathbb{C}(t)$.)

An antiderivative extension $K \supset F$ with no new constants is a Liouville extension [M, p.80] and any Picard–Vessiot extension $E \supset F$ which embeds in $K \supset F$ is then seen to be itself an antiderivative extension with unipotent differential Galois group. Conversely, a Picard–Vessiot extension with unipotent differential Galois group is seen to be an antiderivative extension, so that "Picard–Vessiott antiderivative extension" is the same as "Picard–Vessiot extension (or differential Galois extension) with unipotent differential Galois group".

Next, we will consider differential extensions of F which are unions of their Picard–Vessiot subextensions. These are called infinite Picard–Vessiot extensions (although they in fact may be Picard–Vessiot extensions in the ordinary sense): the differential field E is an infinite Picard–Vessiot of the differential field F if $E = \cup_\alpha E_\alpha$ where each E_α is a Piacrd–Vesiot extension of F. We will show that there exist infinite Picard–Vessiot extensions of F which contain copies of all Picard–Vessiot extensions. This applies to antiderivative Picard–Vessiot extensions as well, and such an infinite Picard–Vessiot antiderivative extension of F is called a Picard–Vessiot antiderivative closure of F. It is shown to be unique up to isomorphism.

The group of differential automorphisms of an infinite Picard–Vessiot extension carries the structure of a pro–affine algebraic group and the group of an infinite Picard–Vessiot antiderivative extension is prounipotent. This applies in particular to the group of the Picard–Vessiot antiderivative closure.

In the special case that $F = \mathbb{C}(t)$ with $D = \frac{d}{dt}$, the "lifting problem" for solvable, and in particular unipotent, differential Galois groups always has a solution. This implies that the differential automorphism group G of the Picard–Vessiot antiderivative closure E of $\mathbb{C}(t)$ has no unsplit extensions by $\mathbb{G}_a$. A prounipotent group with this property must be free prounipotent and thus G is free prounipotent. This also implies that E is differentially isomorphic to the function field $F(G)$ of G over F as a G- module. Explicit generators for $F(G)$ are known. The solutions of any linear differential equation over $\mathbb{C}(t)$ which is completely solvable by repeatedly taking antiderivatives appear in E, and hence can be expressed in terms of these explicit generators. The implications of this observation will be the subject of a subsequent paper.

A Picard–Vessiot antiderivative closure can have proper Picard–Vessiot antiderivative extensions. (For $F = \mathbb{C}(t)$, this is a consequence of the example mentioned above.) It will be shown that any differential automorphism of a differential field extends to a differential automorphism of a Picard–Vessiot antiderivative closure. Thus in a finite tower of such closures starting from F, the differential automorphism group of the top field has a normal series where the successive quotients are the prounipotent groups of automorphisms of the various Picard–Vessiot antiderivative closure layers. We look in particular at the group arising in the case of the two step tower for $F = \mathbb{C}(t)$

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paper to the memory of Shimshon Amitsur.

Infinite Picard–Vessiot extensions: Galois theory and closures

This section begins with general material on the Galois theory of infinite Picard–Vessiot extensions and their associated proaffine algebraic groups which will be needed below. J. Kovacic [K] has established the Galois theory of infinite differential field extensions, where the associated Galois groups are proalgebraic but not necessarily proaffine, and the situation considered here is just a special case of that theory. Consequently, we will just sketch the main results, beginning with the statement of the fundamental theorem:

Fundamental Theorem for Infinite Picard–Vessiot Extensions. *Let $E \supset F$ be an infinite Picard–Vessiot extension. Then $G = G(E/F)$ has a canonical structure of proaffine algebraic group and there is a one to one lattice inverting correspondence between differential subfields K, $E \supset K \supset F$, and Zariski closed subgroups H of G given by $K \mapsto G(E/K)$ and $H \mapsto E^H$. If K is itself an infinite Picard–Vessiot extension of F, then the restriction map $G \to G(K/F)$ is a surjection with kernel $G(E/K)$. If H is normal in G, then K^H is an infinite Picard–Vessiot extension of F.*

Proof. We let $\mathcal{S} = \mathcal{S}(E/F)$ stand for the set of all ordinary Picard–Vessiot subextensions of F in E. Since automorphisms of a differential field preserve Picard–Vessiot subfields [M, Prop. 3.3, p.24], and $\mathcal{S}$ is a directed set there is an isomorphism $G \mapsto \varprojlim\{G(K/F) \mid K \in \mathcal{S}\}$. Since each $G(K/F)$ carries a canonical linear algebraic group structure, this is a canonical structure of proaffine algebraic group on G. The transition maps in the inverse system are surjections [M, Thm. 6.5, p.77] and it follows for $K \in \mathcal{S}$ that $G \to G(K/F)$ is surjective [HM]. This in turn implies that if $\mathcal{S}'$ is any directed set of ordinary Picard–Vessiot subextensions whose union is E then $G = \varprojlim\{G(K/F) \mid K \in \mathcal{S}'\}$.

Now let K_0 be any differential subfield of E containing F. If $K \in \mathcal{S}$ is a Picard–Vessiot extension of F for the linear operator L [M, Def. 3.2 p.24], and $V = L^{-1}(0) \subset K$, then $K_0\langle V\rangle \subset E$ is a Picard-Vessiot extension of K_0 for L as well. Since $K = F\langle V\rangle$, $K_0\langle V\rangle$ is the differential compositum of K_0 and K which we abbreviate K_0K. Since E is the union of $\{K_0K \mid K \in \mathcal{S}\}$, E is an infinite Picard–Vessiot extension of K_0 also, so that $G(E/K_0)$ is a proaffine algebraic group. If we let $\mathcal{S}' \subset \mathcal{S}(E/K_0)$ be $\{K_0K \mid K \in \mathcal{S}\}$, then as remarked above, we have $G(E/K_0) = \varprojlim\{G(K_0K/F) \mid K \in \mathcal{S}\}$, so that $G(E/K_0) \to G(E/K)$ is the inverse limit of the restriction maps $G(K_0K/K_0) \to G(K/F)$ which are morphisms of algebraic groups (see below). In particular, $G(E/K_0)$ is closed in $G(E/K)$.

Since $E^{G(E/F)} \cap K = K^{G(K/F)} = F$ for $K \in \mathcal{S}$, we have $E^{G(E/F)} = F$. Applying this to $E \supset K_0$ shows that the map $K_0 \mapsto G(E/K_0)$ in the Galois correspondence is injective. If H is a proper Zariski closed subgroup of $G(E/F)$, then for some $K \in \mathcal{S}$ the image P of H in the projection $G(E/F) \to G(K/F)$ is proper. It follows that $E^H \cap K = K^P$ which is not F. Thus if $H \leq G(E/F)$ is closed and $E^H = F$ then $H = G(E/F)$. Applying this to $E \supset E^H$ shows that the map $H \mapsto E^H$ in the Galois correspondence is injective.

For $K \in \mathcal{S}$, $G(E/K)$ is the kernel of $G(E/F) \to G(K/F)$. It follows that if $H \leq G(E/F)$ is Zariski closed then $H = \cap\{HG(E/K) \mid K \in \mathcal{S}\}$. E^H contains $\{E^{HG(E/K)} \mid K \in \mathcal{S}\}$. This set is directed and its union M is a differential subfield.

If $\sigma \in G(E/F)$ is trivial on M, then it is trivial on $E^{HG(E/K)}$ and hence belongs to $HG(E/K)$ and then to the intersection over all $K \in \mathcal{S}$, namely H. So $G(E/M) = H = G(E/E^H)$ and thus $M = E^H$. Now suppose H is normal, so that for $K \in \mathcal{S}$ the image P of H in $G(K/F)$ is normal. Then $E^{HG(E/K)} = K^P$ is an ordinary Picard–Vessiot extension of F in K, and it follows that E^H is an infinite Picard–Vessiot extension of F in E. If K_0 is any infinite Picard–Vessiot extension of F in E then $G(K_0/F) = \varprojlim\{G(K/F) \mid K \in \mathcal{S}(K_0/F)\}$, and since $\mathcal{S}(K_0/F) \subseteq \mathcal{S}(E/F)$ it follows that the restriction map $G(E/F) \to G(K_0/F)$ is surjective; the kernel is $G(E/K_0)$. This completes the proof of the Fundamental Theorem.

In the course of the proof, we used the fact (in the notation of the proof) that the restriction maps $G(K_0K/K_0) \to G(K/F)$ are injective morphisms of algebraic groups. This follows because $K_0K = K_0\langle V\rangle$ and $K = F\langle V\rangle$ where V is the (full) set of solutions of $L = 0$ in E. Restriction to V provides the vertical maps in the following commutative diagram, which are morphisms of algebraic groups:

$$\begin{array}{ccc} G(K_0K/K_0) & \longrightarrow & G(K/F) \\ \downarrow & & \downarrow \\ GL(V) & = & GL(V) \end{array}$$

Thus the restriction map in question is a morphism, and in particular has closed image.

Next, we want to consider infinite Picard–Vessiot extensions of F which contain copies of all Picard–Vesiot extensions of F. A construction is offered in [M, p.38], starting with a maximal no new constants extension of F. But if $E \supset F$ is any extension whose constants coincide with those of F (a no new constants extension), and S is any set of differential indeteminates, then the differential field $E\langle S\rangle$ is also a no new constants extension of F, so no maximal no new constants extension can exist.

We provide an alternative construction. To begin, we note that the isomorphism classes of Picard–Vessiot extensions of F are a set of cardinality at most that of F: let $\mathcal{L}$ be the set of all homogeneous linear differential operators over F. Then $\mathcal{L}$ and F have the same cardinality. For each $L \in \mathcal{L}$, let $K_L \supset F$ be a Picard–Vessiot extension for L. Then any Picard–Vessiot extension of F must be isomorphic to some K_L. Thus the set of isomorphism classes of Picard–Vessiot extensions of F is indeed a set and of cardinality at most that of $\mathcal{L}$, hence at most that of F.

We introduce the following notation: let $\mathcal{A}$ denote the set of isomorphim classes of Picard–Vessiot extensions of F.

For each $\alpha \in \mathcal{A}$ we select a Picard–Vessiot extension $E_\alpha \in \mathcal{A}$. Let $S(\mathcal{A}) = \otimes_F\{E_\alpha \mid \alpha \in \mathcal{A}\}$, regarded as a differential ring over F, and for any subset $\mathcal{B} \subseteq \mathcal{A}$ we let $S(\mathcal{B}) \subseteq S(\mathcal{A})$ be the subalgebra generated by $\{E_\alpha \mid \alpha \in \mathcal{B}\}$.

To motivate the following construction, suppose that K is an infinite Picard–Vessiot extension of F and that $\{K_i \mid i \in I\}$ is the set all Picard–Vessiot subextensions of F in K. Since all the differential homomorphisms from a Picard–Vessiot extension to a no new constants have the same image, the K_i belong to distinct isomorphism classes. Let $\alpha(i)$ denote the isomorphism class of K_i and let $\alpha(I) \subseteq \mathcal{A}$ denote the set of all such. Since $K = \cup\{K_i \mid i \in I\}$, it follows that K is a differential homomorphic image of $S(\alpha(I))$.

We consider the set $\mathcal{S}$ of pairs $(\mathcal{B}, M)$ where $\mathcal{B} \subseteq \mathcal{A}$ and M is a differential ideal of $S(\mathcal{B})$ such that $S(\mathcal{B})/M$ is an infinite Picard–Vessiot extension of F. We partially order $\mathcal{S}$ by $(\mathcal{B}_1, M_1) \leq (\mathcal{B}_2, M_2)$ if $\mathcal{B}_1 \subseteq \mathcal{B}_2$ and $M_2 \cap S(\mathcal{B}_1) = M_1$.

Let $\mathcal{C} = \{(\mathcal{B}_j, M_j)\}$ be a chain in S. Let $\mathcal{B} = \cup_j \mathcal{B}_j$ and let $M = \cup_j M_j$. Then $S(\mathcal{B})$ is the (directed) union (direct limit) of the $S(\mathcal{B}_j)$ and M is a differential ideal of $S(\mathcal{B})$. Moreover, $M \cap S(\mathcal{B}_j) = M_j$. It follows that $S(\mathcal{B})/M$ is the direct limit of the infinite Picard–Vessiot extensions $S(\mathcal{B}_j)/M_j$, and hence is itself an infinite Picard–Vessiot extension. It follows that $(\mathcal{B}, M)$ is in $\mathcal{S}$ and is an upper bound for $\mathcal{C}$.

Thus $\mathcal{S}$ has a maximal element $(\mathcal{A}_0, M_0)$. We claim that $\mathcal{A}_0 = \mathcal{A}$. For suppose not, and let α be in $\mathcal{A} - \mathcal{A}_0$. Let L be a linear differential operator over F for which E_α is a Picard–Vessiot extension of F. Let $K = S(\mathcal{A}_0)/M_0$ and let K_1 be a Picard–Vessiot extension of K for L. Let $V = L^{-1}(0) \subset K_1$. Then the differential subfield $E = F\langle V \rangle$ of K generated by V over F is a Picard–Vessiot extension of F for L, hence isomorphic to E_α. K_1 is a no new constants extension of F, and so it follows that the union K_0 of all the Picard–Vessiot subextensions of F in K_1 is an infinite Picard–Vessiot extension of F. We further have that K_0 properly contains K (since E is contained in the former but not the latter). Let $\mathcal{B}$ be the set of isomorphism classes of Picard–Vessiot extensions of F in K_1 which are not contained in K. It follows that K_0 is a homomorphic image of $K \otimes S(\mathcal{B})$, and hence of $S(\mathcal{A}_0 \cup \mathcal{B})$, and that the kernel M of this homomorphism intersects $S(\mathcal{A}_0)$ in M_0. But then $(\mathcal{A}_0, M_0) < (\mathcal{A}_0 \cup \mathcal{B}, M)$, which contradicts the maximality of $(\mathcal{A}_0, M_0)$.

It follows that $\mathcal{A}_0 = \mathcal{A}$ and hence that $S(\mathcal{A})/M_0$ is an infinite Picard–Vessiot extension of F which contains a representative of every isomorphism class of Picard–Vessiot extension of F.

This proves the existence part of the following theorem:

Theorem. *There is an infinite Picard–Vessiot extension E_0 of F which contains an isomorphic copy of every Picard–Vessiot extension of F and is the unique (up to isomorphism) infinite Picard–Vessiot extension of F with this property.*

Proof. The field $S(\mathcal{A})/M$ just constructed is an infinite Picard–Vessiot extension which contains an isomorphic copy of every Picard–Vessiot extension of F. The proof of uniqueness follows as in [M, Thm. 3.31, p.38].

A field E_0 of the type in the theorem is called a *complete Picard–Vessiot compositum* for F [M, Def. 3.32, p.39]. The proaffine algebraic group $G(E_0/F)$ is called the *Picard–Vessiot differential Galois group* of F. If E_1 is another complete Picard–Vessiot compositum for F and $\tau : E_0 \to E_1$ is a differential isomorphism over F then $G(E_1/F) = \tau G(E_0/F)\tau^{-1}$ so $G(E_0/F)$ and $G(E_1/F)$ are isomorphic as abstract groups.

We have the following extension property:

Proposition. *If $K_0 \supset F$ is any infinite Picard–Vessiot extension then there is a complete Picard–Vessiot compositum E_0 of F such that $E_0 \supset K_0 \supset F$. Then by the Fundamental Theorem, $G(K_0/F)$ is a homomorphic image of $G(E_0/F)$.*

Proof. Let $E_1 \supseteq K_0$ be a complete Picard–Vessiot compositum for K_0. Each linear operator L over F, as an operator over K_0 has a Picard–Vessiot extension in E_1. If $W = L^{-1}(0) \subset E_1$, then $E(L) = F\langle W \rangle$ is a Picard–Vessiot extension of F for L contained in E_1. Now E_1 is a no new constants extension of F, and hence the union E_0 of the Picard–Vessiot extensions of F in E_1 is an infinite Picard–Vessiot extension of F by [M, Prop. 3.30, p. 38]. Both K_0 and $E(L)$ are included in E_0. Since L is arbitrary, it follows that E_0 is a complete Picard–Vessiot compositum for F.

In general, an automorphism of F does not lift to a Picard–Vessiot extension of F, of course. (This is what prevents a Picard–Vessiot extension of a Picard–Vessiot extension from being itself Picard–Vessiot, in general.). However, this is true for the complete Picard–Vessiot compositum:

Theorem. *Let E_0 be a complete Picard–Vessiot compositum of F, and let σ be a differential automorphism of F. Then there is a differential automorphism Σ of E_0 such that Σ coincides with σ on F.*

Proof. Let $K \subset E_0$ be a Picard–Vessiot extension for the operator $L = Y^{(n)} + a_{n-1}Y^{(n-1)} + \dots a_0Y^{(0)}$ and let L^σ be the operator obtained from applying σ to the coefficients of L: $L^\sigma = Y^{(n)} + \sigma(a_{n-1})Y^{(n-1)} + \dots \sigma(a_0)Y^{(0)}$. K can be constructed up to isomorphism by taking a full universal solution algebra for L ($R(L) = F[y_j^{(i)}][w^{-1}]$, $0 \leq i \leq n-1$, $1 \leq j \leq n$ with derivation $D(y_j^{(i)}) = y_j^{(i+1)}$ for $i < n-1$ and $D(y_j^{(n-1)}) = -\Sigma_0^{n-1} a_i y_j^{(i)}$; $w = \det(y_j^{(i)})$), moding out a maximal differential ideal P, and forming the quotient field $E(L) = Q(R(L)/P)$. [M, Thm. 3.34, p.25]. We can similarly construct a full universal solution algebra $R(L^\sigma)$ for L^σ over $\sigma(F)$; its derivation D_σ satisfies $D_\sigma(y_j^{(i)}) = y_j^{(i+1)}$ for $i < n-1$ and $D_\sigma(y_j^{(n-1)}) = -\Sigma_0^{n-1}\sigma(a_i)y_j^{(i)}$.

We can define an isomorphism $\hat{\sigma} : R(L) \to R(L^\sigma)$ by $a \mapsto \sigma(a)$ for $a \in F$ and $y_j^{(i)} \mapsto y_j^{(i)}$ for all i, j. One can verify that $\hat{\sigma}$ is a differential isomorphism, and by definition it extends σ. Let P^σ be $\hat{\sigma}(P)$. Then $\hat{\sigma}$ passes to an isomorphism $R(L)/P \to R(L^\sigma)/P^\sigma$ and then to an isomorphism of quotient fields $\overline{\sigma} : E(L) \to E(L^\sigma)$ which extends $\sigma : F \to \sigma(F)$. (Note that we have not used that $\sigma(F) = F$ in the construction of $\overline{\sigma}$.) Preceding $\overline{\sigma}$ with an F isomorphism $K \to E(L)$ and following it with an F embedding $E(L^\sigma) \to E_0$ then gives a map $\tau : K \to E_0$ which fits into a commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{\tau} & E_0 \\ \uparrow & & \uparrow \\ F & \xrightarrow{\sigma} & F \end{array}$$

Next, we consider the set of all pairs (K_0, τ_0) where $K_0 \subset E_0$ is an infinite Picard–Vessiot extension of F and

$$\begin{array}{ccc} K_0 & \xrightarrow{\tau_0} & E_0 \\ \uparrow & & \uparrow \\ F & \xrightarrow{\sigma} & F \end{array}$$

commutes, which we partially order by the relation $(K_0, \tau_0) \leq (K_1, \tau_1)$ if $K_0 \subseteq K_1$ and τ_1 restricts to τ_0 on K_0. Suppose (K_0, τ_0) is a maximal element of this set (which exists by Zorn's lemma). If $K_0 \neq E_0$, then there is a Picard-Vessiot extension K of F in E_0 not contained in K_0. Let K_1 be the differential composite

K_0K, which is a Picard–Vessiot extension of K_0: if K is a Picard–Vessiot extension of F for L, then, as in the Fundamental Theorem, K_0K is a Picard– Vessiot extension of K_0 for L. Using full universal solution algebras over K_0 we can, as above, extend $\tau_0 : K_0 \to \tau_0(K_0)$ to Picard–Vessiot extensions $E^0(L)$ of K_0 and $E^0(L^\sigma)$ of $\tau_0(K_0)$ for L and L^σ constructed from full universal solution algebras: $\overline{\tau_0} : E^0(L) \to E^0(L^\sigma)$. We let K^σ denote the Picard–Vessiot extension of F in E_0 for L^σ. Then the diferential compositum $\tau_0(K_0)K^\sigma$ is a Picard–Vessiot extension of K_0 for L^σ. So if we precede $\overline{\tau_0}$ with an isomorphism of $E^0(L)$ with K_0K and follow it by an isomorphism of $E^0(L^\sigma)$ with $\tau_0(K_0)K^\sigma$ then the result is an embedding $\tau_1 : K_1 = K_0K \to E$ extending τ_0, and hence a pair (K_1, τ_1) strictly larger than the maximal pair (K_0, τ_0). Since no such pair exists, we conclude that $K_0 = E_0$ and that τ_0 is a differential automorphism Σ of E_0 extending σ.

The automorphism Σ of the theorem does not, in general, preserve Picard–Vessiot subextensions of F in the complete compositun E_0. However, it does carry them to Picard–Vessiot extensions, as the following lemma shows:

Lemma. *Let $K_1 \supset F$ and $K_2 \supset F$ be differential field extensions, and let $\tau : K_1 \to K_2$ be a differential isomorphism that carries F to itself. Suppose K_1 is a Picard–Vessiot extension of F. Then so is K_2, and $G(K_2/F) = \tau G(K_1/F)\tau^{-1}$.*

Proof. Let $G = G(K_1/F)$. There is a subset V_1 of K_1 which is a finite dimensional vector space over C, generates K_1 over F as a differential field, and satisfies $G(V_1) = V_1$. Let $V_2 = \tau(V_1)$. V_2 is a finite dimensional C- vector space which generates K_2 as a differential field over F. Let $H = \tau G \tau^{-1}$. Then $H(V_2) = V_2$. Moreover H is a group of differential automorphisms of K_2 which is the identity on F, and $K_2^H = F$. Then by [M, Prop. 3.9, p.27], K_2 is a Picard–Vessiot extension of F, and by [M, Thm. 6.5, p.77], $H = G(K_2/F)$.

Since infinite Picard–Vessiot extensions are unions of ordinary Picard–Vessiot extensions, the lemma also implies the following form of the infinite case:

Corollary. *Let E be an infinite Picard–Vessiot extension of F and let $E \supset K$ be an infinite Picard–Vessiot subextension. Let τ be a differential automorphism of E which carries F to itself. Then $\tau(K)$ is also an infinite Picard–Vessiot subextension and $G(\tau(K)/F) = \tau G(K/F)\tau^{-1}$.*

We want to apply this in the next section to a construction (the antiderivative closure) related to the complete Picard–Vessiot compositum. The connection will be through the following proposition:

Proposition. *Suppose $K \supset F$ is an infinite Picard–Vessiot extension. Let $G = G(K/F)$, let $\overline{G}$ be the maximal prounipotent quotient of G and let $H \leq G$ be the kernel of $G \to \overline{G}$. Suppose τ is a differential automorphism of E which carries F to itself. Then $\tau(K^H) = K^H$.*

Proof. K^H is an infinite Picard–Vessiot extension of F by the Fundamental Theorem. Thus $K^H = \cup\{K_i \mid i \in I\}$ where each K_i is an ordinary Picard–Vessiot extension of F, and $\overline{G} = \varprojlim\{G(K_i/F) \mid i \in I\}$. Also $G(K^H/F) \to G(K_i/F)$ is surjective for every $i \in I$, and so $G(K_i/F)$ is unipotent for every $i \in I$. As we just observed in the corollary, $\tau(K^H)$ is also an infinite Picard–Vessiot extension. In fact $\tau(K^H) = \cup\{\tau(K_i) \mid i \in I\}$ and $G(\tau(K^H)/F) = \varprojlim\{G(\tau(K_i)/F) \mid i \in I\}$. For each i, $G(\tau(K_i)/F) = \tau G(K_i/F)\tau^{-1}$. A linear algebraic group is unipotent if and only if it is nilpotent and has no elements of finite order. Since these group theoretic conditions are preserved by conjugation, we have that each $G(\tau(K_i)/F)$ is unipotent and hence $G(\tau(K^H)/F)$ is prounipotent. By construction, this means that the kernel $G(E/\tau(K^H))$ of $G \to G(\tau(K^H)/F)$ must contain H. By the Fundamental Theorem, this implies that $\tau(K^H) \subseteq K^H$.The same argument applied to τ^{-1} shows that $\tau(K^H) = K^H$.

Anti–derivative extensions and closure

We begin with some elementary facts about anti-derivative extensions.

Proposition. *Let $E \supset F$ be an antiderivative extension generated by the single element y. Then E is generated as a field over F by y. If $y \notin F$, then y is transcendental over F. $E \supset F$ is a Picard–Vessiot extension if and only if it is a no new constants extension, and if it is Picard–Vessiot then $G(E/F)$ is isomorphic to $\{e\}$ or $\mathbb{G}_a$.*

Proof. If $y' \in F$ then the subfield $F(y)$ is a differential subfield of E so if y generates E as a differential field over F, it generates it as a field. Then y is either in F or transcendental over it [M, Rem. 1.10.2, p.7]. Picard–Vessiot extensions have no new constants. Conversely, if $E = F(y)$ has no new constants, and $y \notin F$ then $y' = a \neq 0$. Then 1 and y are a full set of solutions to $L = Y^{(2)} - \frac{a'}{a}Y^{(1)} = 0$ so $F(y)$ is a Picard–Vessiot extension of F for L. In this case $G(E/F) = \mathbb{G}_a$ [M, Ex. 4.24, p.54], and of course if $y \in F$ then $G(E/F)$ is trivial.

Corollary. *Let $E \supset F$ be an antiderivative extension. Then there exist elements $x_1, \dots, x_m$ in E such that for each i we have $x_i' \in F(x_1, \dots, x_{i-1})$ but $x_i \notin F(x_1, \dots, x_{i-1})$.*

Proof. Apply the proposition to a set of elements $y_1, \dots y_n$ as in the definition of antiderivative extension, and discard those that give trivial intermediate extensions.

By [M, Ex. 5.23, p.71] a Picard–Vessiot extension with differential Galois group isomorphic to $\mathbb{G}_a$ is an antiderivative extension generated by a single element. The following is a generalization of this to finitely generated antiderivative extensions.

Proposition. *Let $E \supset F$ be a finitely generated extension. The following are equivalent:*

(1) *E is a Picard–Vessiot antiderivative extension of F.*
(2) *E is a Picard–Vessiot subextension of an antiderivative extension of F.*
(3) *E is a Picard–Vessiot extension with unipotent differential Galois group*

Proof. Obviously the first condition implies the second. And since unipotent groups have normal series with $\mathbb{G}_a$ factors the fact that these factor extensions are singly generated antiderivative extensions shows that the third condition implies the first. So we assume that $E \supset F$ is a Picard–Vessiot extension and is a subextension of the antiderivative extension $K \supset F$. The corollary shows that $K = F(x_1, \dots, x_m)$ is a Liouville extension. Now we repeat the argument of [M, Thm. 6.8, p.80]: we prove the third condition by induction on m. Let $F_2 = F(x_1)$ and consider $K \supset EF_2 \supset F_2$. By induction, $G(EF_2/F_2)$ is unipotent, and it injects by restriction into $G(E/F)$ with image $G(E/E \cap F_2)$. (We use the remark above to see that the image is closed.)

We saw above that $F_2 \supset F$ is Picard–Vessiot with group $\mathbb{G}_a$, which means that $E \cap F_2$ is either F_2 or F, and in any case is Picard–Vessiot over F. In the exact sequence

$$1 \to G(E/E \cap F_2) \to G(E/F) \to G(E \cap F_2/F) \to 1$$

$G(E \cap F_2/F)$ is either $\mathbb{G}_a$ or trivial, hence unipotent, and since $G(E/E \cap F_2)$ is also unipotent, so is $G(E/F)$.

An analysis of the proof of the proposition shows, in the notation of the proposition, that if $E \supset F$ is a Picard–Vessiot extension contained in the antiderivative extension $K = F(x_1, \dots x_m) \supset F$ then the unipotent group $G(E/F)$ has dimension at most m.

One can restate the conclusions of the proposition by saying that a full set of solutions of a monic linear homogeneous differential operator over F may be obtained by repeated adjunctions of antiderivatives (that is, a Picard–Vessiot extension E for L over F is embedded in an antiderivative extension) if and only if the differential Galois group of a Picard–Vessiot extension for L over F is unipotent, in which case the Picard–Vessiot extension for L is itself an antiderivative extension. It is natural to call such extensions a Picard–Vessiot antiderivative extension.

We will show later that an antiderivative extension need not be Picard–Vessiot.

In analogy with complete Picard–Vessiot composita, we shall consider complete Picard–Vessiot antiderivative composita: this is an infinite Picard–Vessiot extension $K_0 \supset F$ such that K_0 is the union of its Picard–Vessiot subextensions with unipotent differential Galois group, and K_0 contains a copy of every Picard–Vessiot antiderivative extension of F. We simplify terminology a bit and call such an extension a *Picard-Vessiot Antiderivative Closure* (abbreviation: *PVAC*) of F.

It is easy to establish existence and basic properties from similar facts for the complete compositum:

Theorem. *let $E_0 \supset F$ be a complete Picard–Vessiot compositum, and let H be the minimal closed normal subgroup of $G(E_0/F)$ such that $G(E_0/F)/H$ is prounipotent. Then $K_0 = E_0^H$ is a Picard–Vessiot antiderivative closure of F, and any Picard–Vessiot antiderivative closure of F is isomorphic to K_0. If σ is any differential automorphism of F, there is a differential automorphism Σ of K_0 whose restriction to F is σ.*

Proof. By the Fundamental Theorem, K_0 is an infinite Picard–Vessiot extension with differential Galois group $\overline{G} = G(E_0/F)/H$. Since $\overline{G}$ is prounipotent, every Picard–Vessiot extension K of F contained in K_0 has $G(K/F)$ unipotent. Now let $E \supset F$ be any Picard–Vessiot extension with unipotent differential Galois group. E embeds in E_0 and we may assume that E is actually a subextension of E_0. Suppose that under the restriction surjection $G = G(E_0/F) \to G(E/F)$ (which has kernel $G(E_0/E)$) H has non-trivial image. Then $H \cap G(E_0/E)$ is a proper subgroup of H. But this intersection is the kernel of the diagonal map $G \to \overline{G} \times G(E/F)$ whose range is a prounipotent group. This contradicts the minimality of H, so H has trivial image in $G(E/F)$, which implies that $E \subset E_0^H = K_0$. Thus K_0 contains a copy of every Picard–Vessiot antiderivative extension of F, and hence is a Picard–Vessiot antiderivative closure.

If K_1 is any Picard–Vessiot antiderivative closure of F, then, since K_1 is in particular an infinite Picard–Vessiot extension, we can embed K_1 in a complete

Picard–Vessiot compositum E_1. Since E_1 is isomorphic to E_0, we can replace K_1 by an isomorphic copy contained in E_0. Then every Picard–Vessiot subextension of K_1, and hence K_1 itself, is a subextension of K_0 by the above argument.

Finally, if σ is any differential automorphism of F, it extends to an automorphism of E_0, and then restricts to an automorphism of K_0, by the theorems in the preceding section.

We complete this section with an example of an antiderivative extension that can't be embedded in a Picard–Vessiot extension. Let $F = \mathbb{C}(t)$ with derivation $\frac{d}{dt}$; let $K = F(y)$ where y is an indeterminant with $y' = \frac{1}{t}$; and let $E = K(z)$ where z is an indeterminant with $z' = \frac{1}{yt}$. ($E = \mathbb{C}(t, y, z)$ can be thought of as $\mathbb{C}(t, \log(t), \log(\log(t)))$.) Then E is a no new constants extension of F: one can see that the polynomial ring $F[y]$ has no differential ideals, for if one existed, by the argument in [M, Remark 1.10.1 p.7] $y' = \frac{1}{t}$ would be a derivative in F. So by [M, Cor. 1.18, p.11] the quotient field K has no new constants. A similar analysis applied to the polynomial ring $K[z]$ shows that since $z' = \frac{1}{yt}$ is not a derivative in K, the quotient field E has no new constants.

Now suppose that $M \supset F$ is a Picard-Vessiot extension with $M \supset E \supset F$. Let $G = G(M/F)$ and let $\sigma \in G$. Since $K \supset F$ is Picard–Vessiot (with group $G(K/F) = \mathbb{G}_a$), $\sigma(y) = y + a$ for some $a \in \mathbb{C}$, and every a occurs for some σ. Since σ is a differential automorphism, $\sigma(z)' = \sigma(z') = \frac{1}{(y+a)t}$. Thus $\sigma(z)$ can be thought of as $\log(\log(t) + a)$.

Suppose $\sigma_1, \ldots, \sigma_n \in G$ are such that $\sigma_i(y) = y + a_i$ with $a_1, \ldots, a_n$ distinct, positive real numbers. Suppose that $\sigma_1(z), \ldots, \sigma_n(z)$ are algebraically dependent over $\mathbb{C}$, say $p(\sigma_1(z), \ldots, \sigma_n(z)) = 0$ for some polynomial p. We write p in terms of the last variable to get an equation like $q_s(\sigma_1(z), \ldots, \sigma_{n-1}(z))\sigma_n(z)^s + \cdots + q_0(\sigma_1(z), \ldots, \sigma_{n-1}(z)) = 0$ with coefficients coming from complex polynomials $q_0. \ldots, q_s$. Then we select a sequence $t_1, t_2, \ldots$ of positive reals so that $\log(t_i)$ converges to $-a_n$. Then $\sigma_n(z)(t_i)$ goes to $-\infty$ while the $q_j(\sigma_1(z)(t_i), \ldots, \sigma_{n-1}(z)(t_i))$ stay bounded and $p(\sigma_1(z)(t_i), \ldots, \sigma_n(z)(t_i)) = 0$, which is impossible.

Since M contains $\log(\log(t)+a)$ for every $a \in \mathbb{C}$, it follows that the transcendence degree of E_0 over $\mathbb{C}$ is infinite. As the transcendence degree over $\mathbb{C}$ of a Picard–Vessiot extension of F is finite, this means there is no such M.

We summarize some of the properties of E in the following proposition:

Proposition. *Let $F = \mathbb{C}(t)$ with derivation $\frac{d}{dt}$, let $K = F(y)$ where $y' = \frac{1}{t}$ and let $E = K(z)$ where $z' = \frac{1}{yt}$. Let K_0 be a Picard–Vessiot antiderivative closure of F and let E_0 be a Picard–Vessiot antiderivative closure of K. Then*

(1) *There is no embedding of E over F into K_0.*
(2) *There is an embedding of K_0 over K into E_0 which is not surjective.*

Proof. E is finitely generated over F as a differential field, so the image of any embedding of E over F in K_0 would be contained in a Picard–Vessiot extension of F, and we just saw the impossibility of this. K_0 is an infinite Picard–Vessiot extension of K with prounipotent differential Galois group, and hence embeds in E_0. But E_0 contains an image of E, however, which must lie outside the image of K_0 by the first point. Thus K_0 is properly embedded in E_0.

The lifting property and free prounipotent groups

Suppose that $K \supset F$ is a Picard–Vessiot antiderivative extension and

$$1 \to \mathbb{G}_a \to G \to G(K/F) \to 1$$

is an extension of unipotent groups. If there is a Picard–Vessiot extension $E \supset F$ containing K such that $G(E/F)$ is isomorphic to G so that the restriction $G(E/F) \to G(K/F)$ is equivalent to the given map $G \to G(K/F)$ then we will say that E *solves the lifting problem* for $G \to G(K/F)$. F has the *lifting property* with respect to extensions of unipotents by $\mathbb{G}_a$ (or just lifting property) if every lifting problem has a solution.

We retain the above notation and we let $\overline{G}$ denote $G(K/F)$. K is isomorphic to the quotient field $F(\overline{G}_F)$ of the integral domain $F[\overline{G}_F] = F \otimes_C C[\overline{G}]$ as a $\overline{G}$-module [M,Cor. 5.29, p.74], and, using this isomorphism, the derivation D_K induces a G-equivariant derivation $\overline{D}$ of $F[\overline{G}_F]$. $\overline{D}$, being G- equivariant and extending D_F, is such that $\overline{D} - D_F \otimes 1$ belongs to $F \otimes_C \text{Lie}(G)$ and hence

$$\overline{D} = D_F \otimes 1 + \sum_i^{n-1} f_i \otimes \overline{D_i}$$

where the $\overline{D_i}$ are a C basis of $\text{Lie}(G)$. As we will see in a lemma below, there is no irreducible f in $F[\overline{G}_F]$ with $\overline{D}(f) = af$ for some $a \in F[\overline{G}_F]$.

We can find a basis $D_1, \dots, D_n$ for $\text{Lie}(G)$ such that under the map $\text{Lie}(G) \to \text{Lie}(\overline{G})$ we have $D_i \mapsto \overline{D_i}$ and $D_n \mapsto 0$.

Suppose further that for any element $h \in F[\overline{G}_F]$ it is possible to find $f_n \in F$ such that the differential equation $Y' = f_n + h$ has no solution in $F[\overline{G}_F]$.[1] Then by [M, Thm. 7.6, p.94] $E = F(G_F)$ with the derivation

$$D = D_F \otimes 1 + \sum_i^{n} f_i \otimes D_i$$

is a Picard–Vessiot extension of F which solves the lifting problem for $G \to \overline{G}$.

Finally suppose F has lots of non-derivatives, in the following sense: for any Picard–Vessiot extension $M \supset F$ and any $h \in M$ there is an element $f \in F$ such that the differential equation $Y' = f + h$ has no solution in E. Then by the above every lifting problem has a solution, and hence F has the lifting property.

It is a consequence of [M, Lemma, p.95] that $F = \mathbb{C}(t)$ has lots of non-derivatives, and hence $\mathbb{C}(t)$ has the lifting property.

For the above argument to be complete, we need the following lemma alluded to in the discussion:

Lemma. *Let U be a unipotent group over C and let D be a U equivariant derivation of $F[U_F]$ extending D_F such that $F(U_F)$ has no new constants. Then there are no elements a, f in $F[U_F]$, f irreducible, with $D(f) = af$.*

Proof. The argument is essentially that of fourth paragraph of the proof of [M, Thm. 7.6, p.94], using induction on the dimension of U. Let $Z = \mathbb{G}_a$ be a

[1]Because G is unipotent, the character χ appearing in the proof of [M, Thm. 7.6, p.94] is trivial.

one dimensional central subgroup of U with quotient $\overline{U}$ and let $u : U \to Z$ be a Z equivariant retraction. Regard u as a C valued function on U. Then $F[U_F] = F[\overline{U}_F][u]$ (polynomial ring), $D(u) = 1$, and since $F[\overline{U}_F] = F[U_F]^Z$ we have $D(F[\overline{U}_F]) \supseteq F[\overline{U}_F]$. Suppose we have a and f with $D(f) = af$. We write f as a polynomial in u with coefficients in $F[\overline{U}_F]$. As in the proof of [M, Thm. 7.6, p.94], one sees that $a, D(u) \in F[\overline{U}_F]$. Then if f has degree n in u, its leading coefficient b_n satisfies $D(b_n) = ab_n$, which by induction applied to $\overline{U}$ implies that b_n is a unit. Then f is replaced by $b_n^{-1}f$ so f is monic and its coefficient b_{n-1} of degree $n-1$ satisfies $D(u) = D(-\frac{b_{n-1}}{n})$. Thus $u - \frac{b_{n-1}}{n}$ is a constant and since $F(U_F)$ has no new constants this implies that $u \in F[\overline{U}_F]$, which is a contradiction.

As a corollary of the lemma and the above discussion, then, we have

Corollary. *$\mathbb{C}(t)$ has the lifitng property with respect to extensions of unipotents.*

$\mathbb{C}(t)$ has the further property that every unipotent algebraic group over $\mathbb{C}$, in fact every connected algebraic group, appears as the differential Galois group of a Picard–Vessiot extension of $\mathbb{C}(t)$ [MS]. A field F with this property is said to have the *(unipotent) inverse Galois property.*

If F is any field with the lifting property, by taking a Picard–Vessiot antiderivative closure $E_0 \supset F$ and applying the Fundamental Theorem to $G(E_0/F)$ we can conclude a lifting property for this prounipotent group:

Proposition. *Let F have the liftng property with respect to prounipotent extensions. Let E_0 be a Picard–Vessiot antiderivative closure of F. Suppose that*

$$1 \to \mathbb{G}_a \to G \to \overline{G} \to 1$$

is an exact sequence of unipotent groups. In addition, suppose there is a surjection $G(E_0/F) \to \overline{G}$. Then it lifts to a homomorphism $G(E_0/F) \to G$.

Proof. Let H be the kernel of $G(E_0/F) \to \overline{G}$. Then by the Fundamental Theorem, $K = E_0^H$ is a Picard–Vessiot extension with $G(K/F) = \overline{G}$. By the lifting property $K \supset F$ embeds in a Picard–Vessiot extension $E \supset F$ with $G(E/F) = G$. We may assume that $E \subset E_0$, and then the restriction $G(E_0/F) \to G(E/F)$ is the desired lifitng.

Now we will show that prounipotent groups that have the property of the preceding proposition are free prounipotent.

Proposition. *Let U be a prounipotent group over the algebraically closed characteristic zero field k, and suppose that for every extension of k unipotent groups*

$$1 \to \mathbb{G}_a \to G \to \overline{G} \to 1$$

such that there is a surjection $U \to \overline{G}$ there is a lifting of the surjection to $U \to G$. Then U is free prounipotent.

Proof. By [LM, Thm. 2.9, p.87], a prounipotent group U is free if and only if $H^2(U, k) = 0$, and by [LM, 1.18, p.82] the hypothesized condition on U implies $H^2(U, k) = 0$.

As an immediate consequence of the proposition and the corollary, we obtain our main result:

Theorem. *Let E_0 be a Picard–Vessiot antiderivative closure of the rational function field $F = \mathbb{C}(t)$. Then the differential Galois group $G(E_0/F)$ is a free prounipotent group.*

The second antiderivative closure

In general, a Picard–Vessiot antiderivative closure E_0 of a differential field F will have proper Picard–Vessiot antiderivative extensions, so we can look at a Picard–Vessiot antiderivative closure E_1 of E_0, then a closure E_2 of E_1, and so forth. We confine our attention to the first two steps $E_1 \supset E_0 \supset F$ and assume that both extensions are proper. Let σ be a differential automorphism of E_1 over F, and let $E \supset F$ be a Picard–Vessiot antiderivative extension contained in E_0. It is clear that $\sigma(E) \supset F$ is a Picard–Vessiot antiderivative extension, and hence also contained in E_0. So if $G(E_1/F)$ denotes the group of differential automorphisms of E_1 over F, there is a restriction map $G(E_1/F) \to G(E_0/F)$. By a theorem above, this map is surjective (automorphisms of E_0 can be extended to automorphisms of its closure E_1) and so we have an exact sequence

$$1 \to G(E_1/E_0) \to G(E_1/F) \to G(E_0/F) \to 1$$

where the groups $G(E_1/E_0)$ and $G(E_0/F)$ are prounipotent. And the surjectivity of $G(E_1/F) \to G(E_0/F)$ shows that

$$E_1^{G(E_1/F)} = (E_1^{G(E_1/E_0)})^{G(E_0/F)} = E_0^{G(E_0/F)} = F$$

In general, for the i^{th} iterated closure E_i, we have a normal series for $G(E_i/F)$ whose factors are prounipotent while $E_i^{G(E_i/F)} = F$.

We saw above an example where $E_1 \neq E_0$ for $F = \mathbb{C}(t)$: we let $K = F(y) = \mathbb{C}(t)(\log t)$ and $E = K(z) = \mathbb{C}(t)(\log t)(\log(\log t))$. Then $K \supset F$ is a Picard–Vessiot antiderivative extension, and hence contained in a Picard–Vesiot antiderivative closure E_0, but E is not embedded in any Picard–Vessiot extension of F. Thus there is no $w \in E_0$ such that $w' = \frac{1}{ty}$. But a Picard–Vessiot antiderivative extension $E_0(w)$ with such an element can be constructed. There is an embedding $E \to E_0(w)$ over F, which we can assume is an inclusion with $w = z$. Then $E_0(z)$ can be embedded (included) in a Picard–Vessiot antiderivative closure E_1 of E_0, and since $E_0 \neq E_0(z)$, $E_1 \neq E_0$.

There is also another Picard–Vessiot antiderivative closure which arises in this situation, namely that of K, which we denote $E_0(K)$. E_0 is a union of Picard–Vessiot antiderivative extensions of F which contain K, from which it follows that E_0 can be embedded over K into $E_0(K)$. Also E is a Picard–Vessiot antiderivative extension of K and embeds into $E_0(K)$ as well. We suppose $E_0(K)$ to be constructed so that both are embeddings are inclusions. We thus have extensions $E_0(K) \supset E \supset K$ as well as $E_0(K) \supset E_0 \supset K$.

Because K is of finite transcendence degree over $\mathbb{C}$, every unipotent group is a differential Galois group over K [MS] and K has plenty of non-derivatives. Thus the arguments above applied to K as a base field show that $G(E_0(K)/K)$ is free prounipotent as well. Also, since $G(E_0/K)$ and $G(E_0(K)/E_0)$ are subgroups of the free prounipotent groups $G(E_0(K)/K)$ and $G(E_0/F)$, they are also free [LM, Cor. 2.10, p.87]. The exact sequence

$$1 \to G(E_0(K)/E_0) \to G(E_0(K)/K) \to G(E_0/K) \to 1$$

then is a non-trivial split exact sequence of free prounipotent groups.

The same remarks (except possibly the non-triviality) obviously apply to any antiderivative extension of F in E_0:

Proposition. *Let E_0 be a Picard-Vessiot antiderivative closure of $F = \mathbb{C}(t)$, and let $M \supset F$ be an antiderivative extension of F in E_0. Let $E_0(M)$ be a Picard–Vessiot antiderivative closure of M which contains E_0. Then*

$$1 \to G(E_0(M)/E_0) \to G(E_0(M)/M) \to G(E_0/M) \to 1$$

is a split exact sequence of free prounipotent groups. If M has an antiderivative extension which cannot be embedded in a Picard–Vessiot extension of F, then $E_0(M) \neq E_0$.

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