Math 2443, Sections 16.1-16.2

Review

These notes will supplement (not replace) the lectures based on Sections 16.1 and 16.2.

Section 16.1

(i) **Double integrals over rectangles:** Let \( R \) be the rectangle given by \([a, b] \times [c, d]\). That is,

\[
R = \{(x, y) \mid a \leq x \leq b, \ c \leq y \leq d\}.
\]

We divide this rectangle into subrectangles by dividing the interval \([a, b]\) into \(m\) equal subintervals \([x_{i-1}, x_i]\) of equal width \(\Delta x = \frac{b-a}{m}\), and dividing \([c, d]\) into \(n\) equal subintervals \([y_{j-1}, y_j]\) of equal width \(\Delta y = \frac{d-c}{n}\). We write \(\Delta A = \Delta x \Delta y\). These subdivisions form rectangles \(R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]\). Let \((x_{ij}^*, y_{ij}^*)\) be a sample point in each \(R_{ij}\). Then, the double integral of \(f\) over \(R\) is defined by

\[
\iint_R f(x, y)dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A
\]

if the limit exists.

(ii) If the limit exists, the function is called **integrable** (on \(R\)).

(iii) **Volume:** If \(f(x, y) \geq 0\), then the volume \(V\) of the solid that lies below the surface \(z = f(x, y)\) and above the rectangle \(R\) is

\[
V = \iint_R f(x, y)dA.
\]

(iv) **Average value:** Let \(A(R)\) be the area of the rectangle \(R\). The average value of \(f\) over \(R\) is given by

\[
f_{av} = \frac{1}{A(R)} \iint_R f(x, y)dA.
\]

Section 16.2

(i) **Iterated integrals:** A double integral of the form

\[
\int_c^d \int_a^b f(x, y)dx dy
\]

can be evaluated by first calculating the inner integral with respect to \(x\). The resultant will be a function of \(y\). Let us write

\[
g(y) = \int_a^b f(x, y)dx.
\]
Then the original integral becomes

\[ \int_c^d \int_a^b f(x, y) \, dx \, dy = \int_c^d g(y) \, dy. \]

In this sense, the given integral is called an *iterated integral*. Of course, we can also try to evaluate

\[ \int_a^b \int_c^d f(x, y) \, dy \, dx \]

in a similar manner. The next theorem tells us that when the \( f \) is continuous, these two ways of finding the double integral yield the same answer.

(ii) **Fubini’s Theorem** If \( f \) is continuous on the rectangle \( R = [a, b] \times [c, d] \), then

\[
\iint_R f(x, y) \, dA = \int_c^d \int_a^b f(x, y) \, dx \, dy = \int_a^b \int_c^d f(x, y) \, dy \, dx.
\]

0.1 **Example:** Calculate the double integral

\[
\iint_R (6x^2y^3 - 5y^4) \, dA
\]

where \( R = \{(x, y) \mid 0 \leq x \leq 3, \ 0 \leq y \leq 1\} \).

**Solution:** By Fubini’s Theorem, we have

\[
\iint_R (6x^2y^3 - 5y^4) \, dA = \int_0^3 \int_0^1 (6x^2y^3 - 5y^4) \, dy \, dx
\]

\[
= \int_0^3 \left( \frac{6}{4} x^2 y^4 - y^5 \right)_0^1 \, dx
\]

\[
= \int_0^3 \left( \frac{3}{2} x^2 - 1 \right) \, dx
\]

\[
= \left( \frac{1}{2} x^3 - x \right)_0^3
\]

\[
= \frac{1}{2} (3^3) - 3 = \frac{21}{2}.
\]

Since the function is continuous, we’ll get the same answer if we calculate

\[
\int_0^1 \int_0^3 (6x^2y^3 - 5y^4) \, dx \, dy.
\]

0.2 **Example:** Calculate the double integral

\[
\iint_R xye^{2y} \, dA, \quad R = [0, 1] \times [0, 2].
\]

**Solution:** By Fubini’s Theorem, we have

\[
\iint_R xye^{2y} \, dA = \int_0^1 \int_0^2 xye^{2y} \, dy \, dx.
\]
Notice that we can’t find the integral \( \int_0^2 xye^{x^2y}dy \) by any direct method and thus we should reverse the order of integration. Thus, we write (using the substituting \( u = x^2y \) in the inner integral)

\[
\int \int_R xye^{x^2y}dA = \int_0^2 \int_0^1 xye^{x^2y}dxdy = \int_0^2 \left( \int_0^1 \frac{1}{2}e^{u}du \right)dy = \frac{1}{2} \int_0^2 (e^y - 1)dy = \frac{1}{2}(e^y - y|_0^1) = \frac{1}{2}(e^2 - 3).
\]

0.3 Example: Find the volume of the solid lying under the elliptic paraboloid \( x^2/4 + y^2/9 + z = 1 \) and above the rectangle \( R = [-1, 1] \times [-2, 2] \).

Solution: The elliptic paraboloid can be expressed as \( z = f(x, y) \) where \( f(x, y) = 1 - x^2/4 - y^2/9 \). The first thing to notice is that \( f(x, y) \geq 0 \) on the rectangle \( R \). Hence, the volume \( V \) under the surface and above the rectangle (by Fubini’s Theorem) is

\[
V = \int \int_R \left( 1 - \frac{x^2}{4} - \frac{y^2}{9} \right) dA = \int_{-1}^1 \int_{-2}^2 \left( 1 - \frac{x^2}{4} - \frac{y^2}{9} \right) dydx
\]

\[
= \int_{-1}^1 \left( y - \frac{x^2y}{4} - \frac{y^3}{27} \right)_{-2}^2 dx
\]

\[
= \int_{-1}^1 \left( 4 - x^2 - \frac{16}{27} \right) dx = \int_{-1}^1 \left( \frac{92}{27} - x^2 \right) dx
\]

\[
= \left( \frac{92}{27}x - \frac{x^3}{3} \right|_1^0 = \frac{184}{27} - \frac{2}{3} = \frac{166}{27}.
\]

0.4 Example: Find the volume of the solid that lies under the plane \( 3x+2y+z = 12 \) and above the rectangle \( R = \{ (x, y) | 0 \leq x \leq 1, -2 \leq y \leq 3 \} \).

Solution: The plane can be expressed as \( z = f(x, y) \) where \( f(x, y) = 12 - 3x - 2y \). Clearly, \( f(x, y) \geq 0 \) on \( R \). Hence, the volume \( V \) under the plane and above the rectangle is

\[
V = \int \int_R (12 - 3x - 2y)dA = \int_0^1 \int_{-2}^3 (12 - 3x - 2y)dydx
\]

\[
= \int_0^1 \left( 12y - 3xy - y^3 \right)_{-2}^3 dx = \int_0^1 (60 - 15x - 5)dx
\]

\[
= \left( 55x - \frac{15}{2}x^2 \right|_0^1 = 55 - \frac{15}{2} = \frac{95}{2},
\]

where we have used Fubini’s Theorem in the first step.
(iii) If \( f(x, y) = g(x)h(y) \), meaning that \( f \) is the product of functions of single variables \( x \) and \( y \) respectively, the Fubini Theorem on \( R = [a, b] \times [c, d] \) gives us

\[
\int_{R} f(x, y) dA = \int_{R} g(x)h(y) dA = \int_{a}^{b} g(x) dx \int_{c}^{d} h(y) dy.
\]

Hence, in such a case, the double integral is simply the product of integrals of \( g \) and \( h \) with respect to \( x \) and \( y \) respectively.

0.5 Example: Find the average value of \( f(x, y) = x^2y \) over the rectangle \( R \) with vertices \((-1, 0), (-1, 5), (1, 5) \) and \((1, 0)\).

**Solution:** We can write \( R = [-1, 1] \times [0, 5] \). Now, \( f \) is the product of \( x^2 \) and \( y \), which are functions of single variables \( x \) and \( y \) respectively. Hence,

\[
\int_{R} x^2 y dA = \int_{-1}^{1} x^2 dx \int_{0}^{5} y dy = \left[ \frac{x^3}{3} \right]_{-1}^{1} \left[ \frac{y^2}{2} \right]_{0}^{5} = \frac{2}{3} \cdot \frac{25}{2} = \frac{25}{3}.
\]

Now, the area of the rectangle \( R \) is \( A(R) = 2 \times 5 = 10 \). Hence the average value of \( f \) over \( R \) is

\[
\bar{f} = \frac{1}{A(R)} \int_{R} x^2 y dA = \frac{1}{10} \cdot \frac{25}{3} = \frac{5}{6}.
\]

(iv) **Remark:** These examples suggest that, in the context of Fubini’s Theorem, both ways of finding the double integral will give you the same answer but in some cases, finding the integral with respect to one variable might be difficult. In such a case it is advisable to change the order of integration. In some other cases, it wouldn’t matter which integral you choose to find first.