Math 2443 Midterm 1

Review

This document contains information that will help you prepare for your first midterm. The syllabus for the test is Sections 15.1 through 15.6. Please be advised that the following is by no means a complete list of things to remember, so you should refer to the class notes and the text.

Section 15.1

(i) Familiarize yourself with the notions of the domain, range, graph and contour map of a function of two variables. Pay attention to the definitions. For a function $f$ of two variables

- the domain $D$ of $f$ is the set of points $(x, y)$ in $\mathbb{R}^2$ for which the function is well-defined. Thus, $D$ is a subset of $\mathbb{R}^2$. Use the notation correctly. For example, the domain $D$ of the function $f(x, y) = \sqrt{9 - x^2 - y^2}$ is $D = \{(x, y) | x^2 + y^2 \leq 9\}$. This represents a disk of radius 3 centered at the origin in $\mathbb{R}^2$.

- the range of $f$ is the set of values $f(x, y)$ that the function can take when $(x, y)$ is in the domain. This is a subset of $\mathbb{R}$. For example, the range of the function defined above is $[0, 3]$. You can express it like so:

$$\text{range } f = \{z | z = \sqrt{9 - x^2 - y^2}, x^2 + y^2 \leq 9\} = [0, 3].$$

So basically, we are looking for the collection of all all possible $z$ when $(x, y)$ is in the domain.

- graph of $f$ is the set of points $(x, y, z)$ such that $z = f(x, y)$ when $(x, y)$ is in the domain of $f$. The graph of a function of two variables is a surface in $\mathbb{R}^3$. In the example above, the graph $f = \{(x, y, z) | z = \sqrt{9 - x^2 - y^2}, x^2 + y^2 \leq 9\} = \{(x, y, z) | x^2 + y^2 + z^2 = 9, z \geq 0\}$ which is a hemisphere of radius 3 centered at $(0, 0, 0)$. To determine the surface corresponding to a particular equation, please refer to Section 13.6 of the book (and the table on page 844 in particular).

The graph of a linear function of two variables is a plane.

- a level curve of $f$ is the set of points $(x, y)$ which satisfy $f(x, y) = k$ where $k$ is a chosen constant. Typically, $k$ is chosen in regular (integer) increments. A contour map (or plot) is a collection of level curves drawn for several values of $k$. In the above example, for each $k$ a level curve is the set of points

$$\{(x, y) | \sqrt{9 - x^2 - y^2} = k\} = \{(x, y) | x^2 + y^2 = 9 - k^2\}$$

which is a single point $(0, 0)$ for $k = 3$ and a circle of radius $\sqrt{9 - k^2}$ for $k = 0, 1, 2$.

(ii) Understand the relationship between level curves and the graph of a function of two variables. If we slice (intersect) the graph of $f$ by the horizontal plane $z = k$, and project the resulting curve of intersection onto the $x - y$ plane, we get the level curve $f(x, y) = k$. The closer the level curves are, the steeper the graph is. Conversely, the farther the level curves are to each other, the flatter the graph is. In our example, the level curves are closer to each other for smaller values of $k$ and farther for larger values. This corresponds to the hemisphere being steeper for lower values of $z$ and flatter as $z$ increases to 3.
(iii) All the concepts are defined in the same way for functions of three variables except that the set of points \((x, y, z)\) such that \(f(x, y, z) = k\) are called *level surfaces* not level curves.

**Section 15.2**

Remember the definition of continuity: A function \(f\) of two variables is continuous at \((a, b)\) if

\[
\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).
\]

We say that \(f\) is continuous on a set \(D\) if \(f\) is continuous at every point \((a, b)\) in \(D\).

In general, the functions you will come across in the course will be continuous.

You won’t be asked any questions directly from this section but it is good to remember how finding the limit of a function of two variables is different from that of a function of one variable.

**Section 15.3**

(i) The idea is simple: for a function \(f\) of two or more variables, the partial derivative of \(f\) with respect to (say) \(x\) is obtained by treating all the other variables as constants and then taking the derivative with respect to \(x\). There are many equivalent notations and any one can be used in a given problem. For example, if \(f\) is a function of two variables and we write \(z = f(x, y)\) then

\[
f_x = f_x(x, y) = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x},
\]

and similarly for the partial derivative with respect to \(y\). This is easily extended to functions of three variables and more.

(ii) The partial derivatives \(f_x\) and \(f_y\) of a function \(f\) of two variables at a given point \((x_0, y_0)\) can be interpreted as follows. Let \(z_0 = f(x_0, y_0)\). Then \(f_x(x_0, y_0)\) is the slope of the tangent to the curve \(C_1\) (obtained by intersecting the surface \(z = f(x, y)\) by the plane \(y = y_0\)) at the point \((x_0, y_0, z_0)\) on the surface \(z = f(x, y)\). Similarly, \(f_y(x_0, y_0)\) is the slope of the tangent to the curve \(C_2\) (obtained by intersecting the surface \(z = f(x, y)\) by the plane \(x = x_0\)) at the point \((x_0, y_0, z_0)\) on the surface \(z = f(x, y)\).

(iii) Implicit differentiation: It is often not easy to write an explicit expression for \(z\) in terms of \(x\) and \(y\). In such a case, what we may have available is an equation involving \(x, y\) and \(z\) with the underlying assumption that \(z\) is expressed implicitly as a function of \(x\) and \(y\). For example, if \(z\) is defined implicitly in terms of \(x\) and \(y\) by the equation

\[
x^2 + y^2 + z^2 = e^{xyz} - 1
\]

then, to find \(\frac{\partial z}{\partial x}\), we simply take the partial derivative of each side with respect to \(x\) to get

\[
2x + 2z \frac{\partial z}{\partial x} = e^{xyz} - 1 (yz + xy \frac{\partial z}{\partial x})
\]

and thus

\[
\frac{\partial z}{\partial x} = \frac{yz e^{xyz} - 1 - 2x}{2z - xye^{xyz}}.
\]
(iv) Since first order partial derivatives are themselves multivariable functions, we can easily find partial derivatives of higher order by successive partial differentiation. For a function of two variables, there are four second-order partial derivatives and eight third order partial derivatives. If you are asked to find, let’s say, \( f_{xyz} \), you’d find \( f_x \), \( f_{xy} \) and \( f_{xyz} \) successively. No other derivatives need to be found in this case.

(v) Remember Clairaut’s Theorem. If \( f_{xy} \) and \( f_{yx} \) are both continuous on a given region in \( \mathbb{R}^2 \), they are equal at each point inside the region. So, if in your calculations these are not the same then you know you’ve made a mistake somewhere.

Section 15.4

(i) Tangent plane: We have seen in Section 15.3 that the partial derivatives of a function \( f \) of two variables at a given point \((x_0, y_0)\) are slopes of tangent lines to curves obtained by intersecting the graph of the function (which is a surface) with planes \( y = y_0 \) and \( x = x_0 \) respectively. The plane that contains these tangent lines is called the tangent plane to the surface at the given point.

(ii) If \( f \) has continuous partial derivatives, an equation of the tangent plane to the surface \( z = f(x, y) \) at the point \((x_0, y_0, z_0)\) (where \( z_0 = f(x_0, y_0) \)) is

\[
z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).
\]

(iii) The linearization of \( f \) at \((x_0, y_0)\) is the function \( L \) defined by

\[
L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).
\]

and the approximation \( f(x, y) \sim L(x, y) \) is called the linear approximation of \( f \) at \((x_0, y_0)\).

(iv) So what’s the relationship between the tangent plane and the linearization as defined above? The tangent plane to the surface at a given point \((x_0, y_0, z_0)\) is the graph of the linearization of the function at \((x_0, y_0)\).

(v) Differentiability: Remember the following Theorem:

If the partial derivatives \( f_x \) and \( f_y \) exist around \((a, b)\) and are continuous at \((a, b)\), then \( f \) is differentiable at \((a, b)\).

Thus, to show that a given function of two variables is differentiable at a given point, we just need to find \( f_x \) and \( f_y \) and observe that they are continuous at the given point.

(vi) Differentials: Another way to use linear approximations is via differentials. Given \( dx \) and \( dy \) as independent variables and \( z = f(x, y) \), we define

\[
dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.
\]

The idea is that, if know the relative change in \( x \) and \( y \), we can find the relative change in \( z \) at a given point via this formula. As before, we can extend this notion and formula to functions of three or more variables. This is useful in solving word problems that deal with errors in measurement of individuals dimensions and you are asked to find the maximum error in measurement of a quantity that depends on all the independent variables (such as the volume or surface area of a box). See Homework #3 for details.
Section 15.5

(i) There are two basic cases of the multivariable chain rule that we have studied. If \( f \) is a differentiable function of \( x \) and \( y \), which are both functions of \( t \), then \( z = f(x, y) \) is a differentiable function of \( t \) and

\[
\frac{dz}{dt} = (\frac{\partial f}{\partial x}) \frac{dx}{dt} + (\frac{\partial f}{\partial y}) \frac{dy}{dt} = (\frac{\partial z}{\partial x}) \frac{dx}{dt} + (\frac{\partial z}{\partial y}) \frac{dy}{dt}.
\]

Recall that \( z \) is a dependent variable, \( x \) and \( y \) are intermediate variables and \( t \) is the independent variable. Hence \( z \) is a function of a single variable, which is why we use the ordinary derivative \( \frac{dz}{dt} \).

Similarly, if \( f \) is a function of \( x \) and \( y \) which themselves are functions of \( s \) and \( t \), we have the following:

\[
\frac{\partial z}{\partial t} = (\frac{\partial f}{\partial x}) \frac{\partial x}{\partial t} + (\frac{\partial f}{\partial y}) \frac{\partial y}{\partial t} \\
\frac{\partial z}{\partial s} = (\frac{\partial f}{\partial x}) \frac{\partial x}{\partial s} + (\frac{\partial f}{\partial y}) \frac{\partial y}{\partial s}.
\]

(ii) We can easily write a Chain Rule for functions with several intermediate and independent variables using a Tree Diagram (refer to the class notes).

(iii) Implicit differentiation revisited: We already know how to find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \) if \( z \) is defined implicitly as a function of \( x \) and \( y \) but we can also use the Chain Rule to solve the problem as follows:

Suppose that \( z \) is defined in terms of \( x \) and \( y \) by the equation \( F(x, y, z) = 0 \) where \( F \) is a function of three variables (so \( x \), \( y \) and \( z \) are independent variables). Then, by Chain Rule, the partial derivative with respect to \( x \) yields

\[
F_x(x, y, z) \frac{\partial x}{\partial x} + F_y(x, y, z) \frac{\partial y}{\partial x} + F_z(x, y, z) \frac{\partial z}{\partial x}.
\]

(0.1)

Since \( \frac{\partial x}{\partial x} = 1 \) and \( \frac{\partial y}{\partial x} = 0 \) (because of the fact that \( x \) and \( y \) are independent variables), we get that

\[
\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)}.
\]

Similarly,

\[
\frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}.
\]

(0.2)

In other words, if you rewrite the governing equation as \( F(x, y, z) = 0 \), the previous two equations will give you the desired partial derivatives. For example, consider again the equation

\[x^2 + y^2 + z^2 = e^{xyz-1}\]
which we looked at in Section 15.3. Define $F(x, y, z) = x^2 + y^2 + z^2 - e^{xyz} - 1$. We calculate

\begin{align*}
F_x(x, y, z) &= 2x - yze^{xyz} - 1 \\
F_y(x, y, z) &= 2y - xze^{xyz} - 1 \\
F_z(x, y, z) &= 2z - yxe^{xyz} - 1
\end{align*}

substituting these in (0.1) and (0.2), we get the same expressions for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ as before.

Section 15.6

(i) Given a function $f$ of $x$ and $y$, the partial derivatives $f_x$ and $f_y$ at a point measure the rate of change of the function with respect to one variable when the other variable is fixed. The directional derivative is a generalization of this notion.

The directional derivative of $f$ at $(x_0, y_0)$ in the direction of the unit vector $u = \langle a, b \rangle$ is defined by

$$D_u f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

whenever the limit exists. Notice that, if $u = \langle 1, 0 \rangle$, we have $D_u f(x_0, y_0) = f_x(x_0, y_0)$ and if $u = \langle 0, 1 \rangle$ then $D_u f(x_0, y_0) = f_y(x_0, y_0)$.

(ii) We typically never use the above definition to find the directional derivative. The following theorem is more useful:

If $f$ is a differentiable function of $x$ and $y$, then $f$ has a directional derivative in the direction of any unit vector $u = \langle a, b \rangle$ and

$$D_u f(x, y) = f_x(x, y)a + f_y(x, y)b.$$ 

Remember that this formula works only if $u$ is a unit vector. If the given vector is not a unit vector, then we must first divide by its magnitude to use this formula.

If $u$ is given to be the unit vector which makes an angle of $\theta$ with the positive $x$-axis, then $u = \langle \cos \theta, \sin \theta \rangle$.

(iii) The gradient of a function $f$ of two variables is a vector function, defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial z}{\partial x} \mathbf{i} + \frac{\partial z}{\partial y} \mathbf{j}.$$ 

We can write $D_u f(x, y) = \nabla f(x, y) \cdot u$. In other words, the directional derivative in the direction $u$ is the scalar projection of the gradient vector onto $u$. As before, we can extend this to functions of three variables.

(iv) If $f$ is a differentiable function of two or three variables, the maximum value of the directional derivative $D_u f(x)$ is $|\nabla f(x)|$ and it occurs when $u$ has the same direction as the gradient vector $\nabla f(x)$. In other words, the maximum rate of change of a given function occurs in the direction of the gradient.

(v) Tangent planes to level surfaces: We know how to write an equation for the tangent space to the surface $z = f(x, y)$ at a point $P(x_0, y_0, z_0)$. We consider a more general situation now. Suppose that a surface $S$ is given as the level surface $F(x, y, z) = k$ of a function of three
variables and we’d like to define the tangent plane to the surface at a point \( P(x_0, y_0, z_0) \). For this, let’s choose a curve \( C \) on \( S \) defined by \( \mathbf{r}(t) = (x(t), y(t), z(t)) \) such that \( \mathbf{r}(t_0) = (x_0, y_0, z_0) \). Since \( C \) lies on the surface \( S \), each point \( (x(t), y(t), z(t)) \) satisfies the equation governing \( S \). Thus,

\[
F(x(t), y(t), z(t)) = k.
\]

Taking the derivative of each side with respect to \( t \) (this is Case 1 of the Chain Rule), we get

\[
F_x \left( \frac{dx}{dt} \right) + F_y \left( \frac{dy}{dt} \right) + F_z \left( \frac{dz}{dt} \right) = 0.
\]

or \( \langle F_x, F_y, F_z \rangle \cdot \mathbf{r}'(t) = 0 \). In other words, we have

\[
\nabla F \cdot \mathbf{r}'(t) = 0.
\]

This is also true at the point \( (x_0, y_0, z_0) \), so we have

\[
\nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0
\]

This means that the gradient vector at the point \( P \) is perpendicular to the tangent vector to any curve \( C \) on \( S \) passing through \( P \). We can define the tangent plane to the level surface \( F(x, y, z) = k \) at \( P(x_0, y_0, z_0) \) as the plane that passes through \( P \) and has normal vector as the gradient vector \( \nabla F(x_0, y_0, z_0) \). An equation of this tangent plane is given by

\[
F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.
\]

The normal line to \( S \) at \( P \) is the line through \( P \) which is orthogonal to the tangent plane. Symmetric equations of this line are given by (assuming that none of the partial derivatives are zero at \( P \)).

\[
\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}.
\]

(vi) When the surface \( S \) is given by the graph of a function of two variables \( f \) (that is, \( z = f(x, y) \)), we can define

\[
F(x, y, z) = f(x, y) - z.
\]

Then, for the level surface \( F(x, y, z) = 0 \), we have

\[
F_x(x_0, y_0, z_0) = f_x(x_0, y_0), \quad F_y(x_0, y_0, z_0) = f_y(x_0, y_0), \quad F_z(x_0, y_0, z_0) = -1.
\]

so our equations for the tangent plane to the level surface becomes

\[
f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0
\]

which exactly gives us the equation we found in Section 15.4. We thus have a generalization of the concept of the tangent space to level surfaces.

(vii) The same ideas hold for functions of two variables. The gradient of the function at a point is orthogonal to the tangent vector to the level curve at that point. This tells us that the direction of maximum increase of the function is perpendicular to the level curve at any given point. Thus, the curve of fastest increase is the curve which is tangent to all level curves.