Instructions

- Explain your answers clearly. Credit will be given only if you provide sufficient justification for your answers.
- Calculators, phones, laptops, gaming consoles and other electronic gadgets are not permitted and should be turned off during the exam.

1. [8 points] In each of the following indicate whether the statement is True or False. Explain your reasoning in one line.
   (a) The function \( f(x, y) = \frac{1}{y} e^{xy} \) has no critical points.
   (b) Every saddle point of a given function of two variables is a critical point.
   (c) The region \( \{(x, y)| 0 < x < 2, 0 \leq y \leq 4\} \) is closed and bounded.
   (d) If \( R = [0, 1] \times [0, 1] \) then
      \[
      \int\int_R \left( \frac{1 + x^2}{1 + y^2} \right) \, dx \, dy = \int_0^1 (1 + x^2) \, dx \int_0^1 \frac{1}{1 + y^2} \, dy.
      \]

Solution:

(a) True. We have \( f_x = e^{xy} \) and \( f_y = \left( \frac{x}{y} - \frac{1}{y^2} \right) e^{xy} \) and since \( e^{xy} \neq 0 \), there are no critical points.

(b) True. A saddle point is a critical point which is neither a local minimum or a local maximum.

(c) False. The region can be enclosed in a circular disk, so it is bounded. It is not closed because not all boundary points are contained in the region.

(d) True. This is a special case of Fubini’s Theorem.
2. [8 points] Find the points on the surface \( z^2 = 1 + 3xy \) which are closest (that is, at the shortest distance) to the origin \((0, 0, 0)\).

**Solution:** Let \( P(x, y, z) \) be a point on the given surface. The distance from the origin to \( P \) is given by \( d(x, y, z) = \sqrt{x^2 + y^2 + z^2} \). Now, since \( P(x, y, z) \) lies on the given surface, we have \( d(x, y, z) = \sqrt{x^2 + y^2 + 1 + 3xy} \). We would like to minimize this function but it is easier to look at the square of the distance function:

\[
f(x, y) = x^2 + y^2 + 3xy + 1
\]

We have \( f_x = 2x + 3y \) and \( f_y = 2y + 3x \), thus the only critical point is \((0, 0)\). Also \( f_{xx} = 2, f_{yy} = 2 \) and \( f_{xy} = 3 \). Thus, \( D = 4 - 9 < 0 \), so we can’t use the second derivative test. However, when \( x = 0, y = 0 \), we have \( z = \pm 1 \) and the points \((0, 0, \pm 1)\) are at a distance of 1 from the origin. Notice also that \( x^2 + y^2 + 3xy = (x+y)^2 + xy \geq 0 \). Hence by inspection, we see that \((0, 0, \pm 1)\) are the points on the surface closest to the origin.
3. [9 points] Find the absolute minimum and maximum values of the function

\[ f(x, y) = x^2 - xy - 3y + y^2 \]

on the closed triangular region with vertices (1, 0), (3, 0) and (3, 2).

**Solution:** The triangular region \( D \) can be described by

\[ D = \{(x, y)| 1 \leq x \leq 3, \ 0 \leq y \leq x - 1\}. \]

Let’s first find critical points of \( f \) on the interior of \( D \). We have \( f_x = 2x - y \) and \( f_y = -x - 3 + 2y \). Thus, the only critical point is \((1, 2)\). However, notice that this point lies outside the region \( D \) so we need not consider it.

Next, we find the extreme values of \( f \) on the boundary of \( D \). Let \( L_1 \) be the segment joining \((1, 0)\) and \((3, 0)\), let \( L_2 \) be the segment joining \((3, 0)\) and \((3, 2)\) and let \( L_3 \) be the segment joining \((1, 0)\) and \((3, 2)\).

On \( L_1 \) we have \( y = 0 \), so the function is \( f(x, 0) = x^2 \), which is increasing. Thus, the minimum and maximum values of \( f \) on \( L_1 \) are 1 and 9 respectively.

On \( L_2 \), we have \( x = 3 \) and the function is \( f(3, y) = y^2 - 6y + 9 \). The maximum and minimum values of the function on \( L_2 \) are 9 and 1.

On \( L_3 \), we have \( y = x - 1 \) and the function is \( f(x, x - 1) = x^2 - 4x + 4 = (x - 2)^2 \). The maximum and minimum values are 1 and 0 respectively.

Hence, the absolute maximum and absolute minimum values of \( f \) on the given region are 9 and 0.
4. [8 points] Use Lagrange multipliers to find the extreme values of the function \( f(x, y) = 2x^3 + y^4 \) subject to \( x^2 + y^2 = 16 \).

**Solution:** Let \( g(x, y) = x^2 + y^2 \). Then, Lagrange Multiplier equations \( (f_x = \lambda g_x, f_y = \lambda g_y, g(x, y) = 16) \) yield

\[
\begin{align*}
6x^2 &= 2\lambda x \\
4y^3 &= 2\lambda y \\
x^2 + y^2 &= 16.
\end{align*}
\]

The first two equations can be written as \( 2x(\lambda - 3x) = 0 \) and \( 2y(\lambda - 2y^2) = 0 \). This means that, either \( x = 0 \) or \( x = \frac{\lambda}{3} \) and \( y = 0 \) or \( y = \pm \sqrt{\frac{\lambda}{2}} \). Since \( x = 0, y = 0 \) does not satisfy the constraint equation, we have the following three cases:

1. \( x = 0, y \neq 0 \). In this case, \( y^2 = \frac{\lambda}{2} \). Substituting this in the constraint equation, we get \( \lambda = 32 \) and thus \( y = \pm 4 \). Thus, \( (0, \pm 4) \) are solutions.

2. \( x \neq 0, y = 0 \). In this case, \( x = \frac{\lambda}{3} \) and substituting in the constraint equation, we get \( \lambda = \pm 12 \), so \( (\pm 4, 0) \) are also solutions.

3. \( x \neq 0, y \neq 0 \). In this case, \( x = \frac{\lambda}{3} \) and \( y = \sqrt{\frac{\lambda}{2}} \). Substituting in the constraint equation, we get

\[
\frac{\lambda^2}{9} + \frac{\lambda}{2} = 16, \quad \text{or} \quad \lambda^2 + 9\lambda - 16 = 0
\]

from which, we get \( \lambda \) and find other solutions as well.

You can now substitute these solutions into the function one-by-one and find maximum and minimum values.
5. [9 points] Find the volume of the solid lying under the surface \( z = y\sqrt{x} \) and above the region \( D \) in the \( xy \)-plane bounded by the parabolas \( y = x^2 \) and \( x = y^2 \).

**Solution:** The region \( D \) is given by

\[ \{(x, y) | 0 \leq x \leq 1, \ x^2 \leq y \leq \sqrt{x}\} . \]

The volume \( V \) of the given solid is

\[
V = \int\int_{D} y\sqrt{x} \, dA = \int_{0}^{1} \int_{x^2}^{\sqrt{x}} y\sqrt{x} \, dy \, dx = \int_{0}^{1} \sqrt{x} \left(\frac{y^2}{2}\right)_{x^2}^{\sqrt{x}} \, dx = \frac{6}{55}.
\]
6. [8 points] Evaluate the double integral by reversing the order of integration

\[ \int_{0}^{1} \int_{\sqrt{y}}^{1} \frac{y e^{x^2}}{x^3} \, dx \, dy. \]

**Solution:** The region \( D \) over which the double integral is being taken can be expressed as

\[ D = \{(x, y) | 0 \leq y \leq 1, \sqrt{y} \leq x \leq 2\}, \]

which is a Type I region. To reverse the order of integration, we write \( D \) as a Type II region. It is given by the region below the parabola \( y = x^2 \) and above the \( x \)-axis. That is,

\[ D = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq x^2\}. \]

Hence,

\[ \int_{0}^{1} \int_{\sqrt{y}}^{1} \frac{y e^{x^2}}{x^3} \, dx \, dy = \int_{0}^{1} \int_{0}^{x^2} \frac{y e^{x^2}}{x^3} \, dy \, dx = \int_{0}^{1} \left( \frac{y^2 e^{x^2}}{2x^3} \right)_{0}^{x^2} \, dx = \frac{1}{2} \int_{0}^{2} x e^{x^2} \, dx = \frac{1}{4} (e^4 - 1). \]
7. [Bonus problem, 8 extra points] Find the volume of the solid tetrahedron bounded by the planes \( x + 2y + z = 2 \), \( x = 2y \), \( x = 0 \) and \( z = 0 \).

**Solution:** The tetrahedron has vertices \((1, \frac{1}{2}, 0)\), \((0, 1, 0)\), \((0, 0, 2)\) and \((0, 0, 0)\). The projection of the solid on the \( xy \) plane is the triangular region \( D \) bounded by line segments joining \((0, 0)\), \((1, \frac{1}{2})\) and \((0, 1)\). We can express \( D \) as

\[
\left\{ (x, y) \mid 0 \leq x \leq 1, \frac{x}{2} \leq y \leq -\frac{x}{2} + 1 \right\}
\]

which is a Type I region. Now, the bounded on \( z \) are \( 2 - x - 2y \) and 0. Hence, the volume \( V \) is

\[
V = \int \int_D \left( \int_0^{2-x-2y} 1 \, dz \right) \, dA = \int_0^1 \int_0^{-\frac{x}{2} + 1} (2 - x - 2y) \, dy \, dx
\]

\[
= \int_0^1 \left( 2y - xy - y^2 \right)_{-\frac{x}{2}}^{\frac{x}{2} + 1} \, dx = \int_0^1 (x - 1)^2 \, dx = \frac{1}{3} (x - 1)^3 \bigg|_0^1 = \frac{1}{3}.
\]