1. [5 points] (a) Determine whether the following sequence converges or not. If it converges, find the limit.

\[ a_n = \cos \left( \frac{\sin n}{1 + \sqrt{n}} \right) . \]

(b) Does the sequence

\[ \left\{ (-1)^n \cos \left( \frac{\sin n}{1 + \sqrt{n}} \right) \right\}_{n=1}^\infty \]

converge? Why or why not?

**Solution:**

(a) Let’s look at \( a_n = \frac{\sin n}{1 + \sqrt{n}} \) first. Since \(-1 \leq \sin n \leq 1\) for all \( n \geq 1 \), we have

\[ 0 \leq |a_n| \leq \frac{1}{1 + \sqrt{n}} \]

Also, \( \lim_{n \to \infty} \frac{1}{1 + \sqrt{n}} = 0 \) and the constant sequence \( \{0\} \) obviously converges to zero, thus by Squeeze Theorem, we have \( \lim_{n \to \infty} |a_n| = 0 \) and then by the Absolute Value Theorem for sequence, we conclude that \( \lim_{n \to \infty} a_n = 0 \). Next, the cosine function is continuous at zero, hence

\[ \lim_{n \to \infty} \cos \left( \frac{\sin n}{1 + \sqrt{n}} \right) = \cos \left( \lim_{n \to \infty} \frac{\sin n}{1 + \sqrt{n}} \right) = \cos 0 = 1. \]

Hence, the given sequence converges.

(b) This sequence does not converge since the limit does not exists. The terms oscillate between -1 and +1.
2. [5 points] Determine whether the sequence

\[ \left\{ \frac{\ln n}{n} \right\}_{n=3}^{\infty} \]

is increasing, decreasing or not monotonic. Is the sequence bounded?

**Solution:** Consider the corresponding function \( f(x) = \frac{\ln x}{x} \). We have

\[ f'(x) = \frac{1 - \ln x}{x^2} \]

which is negative for \( \ln x > e = 2.718 \). Hence, the sequence is decreasing for \( n \geq 3 \). The sequence is thus bounded above by the first term \( \frac{\ln 3}{3} \). Each term is also positive for \( n \geq 3 \) and thus the sequence is bounded below by 0. Another way to see that is to notice that \( \lim_{n \to \infty} \frac{\ln n}{n} = 0 \). The conclusion is that the sequence is bounded.