Math 2433 Homework #4

Solutions

Section 12.3.

Use the Integral Test to determine whether the series is convergent or divergent.

3. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$.

**Solution:** The function $f(x) = x^{-1/5}$ is clearly continuous and positive for $x \geq 1$. Also, $f'(x) = -\frac{1}{5}x^{-6/5}$, so $f$ is decreasing on $[1, \infty)$. To use the Integral Test, we calculate

$$\int_1^{\infty} x^{-1/5} dx = \lim_{t \to \infty} \int_1^{t} x^{-1/5} dx = \lim_{t \to \infty} \frac{5}{4} \left( t^{4/5} - 1 \right) = \infty.$$ 

Since the improper integral diverges, by Integral Test, the given series also diverges.

5. $\sum_{n=1}^{\infty} \frac{1}{(2n+1)^3}$.

**Solution:** Here $f(x) = \frac{1}{(2x+1)^3}$ which is positive and continuous for $x \geq 1$. Also, $f'(x) = -4(2x+1)^{-3}$ so $f$ is decreasing on $[1, \infty)$. We have

$$\int_1^{\infty} \frac{1}{(2x+1)^2} dx = \lim_{t \to \infty} \int_1^{t} \frac{1}{(2x+1)^2} dx = \lim_{t \to \infty} \frac{1}{2} \left( -\frac{1}{2x+3} \right)_1^t = \frac{1}{10},$$

so by Integral Test, the series converges.

8. $\sum_{n=1}^{\infty} \frac{n+2}{n+1}$.

**Solution:** The key here is to observe that

$$f(x) = \frac{x+2}{x+1} = 1 + \frac{1}{x+1}.$$ 

It should now be easy to apply the Integral Test so I am not going to write out the solution (but you have to do so).

Determine whether the series is convergent or divergent.

10. $\sum_{n=1}^{\infty} (n^{-1.4} + 3n^{-1.2})$.

**Solution:** The series is

$$\sum \left( \frac{1}{n^{1.4}} + \frac{3}{n^{1.2}} \right)$$

which is a sum of two convergent $p$-series, so it is convergent.
12. \(1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \ldots\)

Solution: The series is of the form \(\sum \frac{1}{n^{3/2}} = \sum \frac{1}{n^{3/2}}\), so it is convergent since it is a \(p\)-series with \(p > 1\).

15. \(\sum \frac{5-2\sqrt{n}}{n^3}\)

Solution: The series is \(\sum \left(\frac{5}{n^3} - \frac{2}{n^{3/2}}\right)\), which is the difference of two convergent \(p\)-series, so it converges.

16. \(\sum \frac{n^2}{n^3+1}\)

Solution: The function \(f(x) = \frac{x^2}{x^3+1}\) is continuous and positive. Also \(f'(x) = \frac{2x-x^4}{(x^3+1)^2}\) which is negative for \(x \geq 2\), so \(f\) is decreasing. We must calculate

\[\int_2^\infty \frac{x^2}{x^3+1} \, dx\]

This integral can be calculated using the substitution \(u = x^3 + 1\). Once again, even though I haven’t done the calculation, you have to do it.

23. \(\sum \frac{e^{1/n}}{n^2}\)

Solution: I won’t go into the details but for \(f(x) = \frac{e^{1/x}}{x^2}\), we have \(f'(x) = -\frac{(e^{1/x}+2x)}{x^3}\), so the function satisfying the properties needed to use the Integral Test. The improper integral \(\int \frac{e^{1/x}}{x^3} \, dx\) can be calculated using the substitution \(u = \frac{1}{x}\). I will let you fill in the blanks.