Stage I  
Round 1

1. Can you tile the following region with 1 × 2 dominoes? Draw it or say why not.

```
  3
  7
```

2. Which is bigger: \((2)^\sqrt{2}\) or \((\sqrt{2})^2\)?

3. A Pythagorean triple \((a, b, c)\) is a collection of three positive integers such that \(a^2 + b^2 = c^2\). For example \((3, 4, 5)\) and \((5, 12, 13)\) are Pythagorean triples because \(3^2 + 4^2 = 5^2\) and \(5^2 + 12^2 = 13^2\).

Find a Pythagorean triple made up entirely of even numbers.

Stage I  
Round 2: Blitz Round

1. An explorer walks 1 mile south, the 1 mile east, and 1 mile north. He finds himself back where he started and encounters a bear. What color is the bear?

2. Let \(a\) and \(b\) be positive real numbers. What is the smallest possible value of \(\frac{a}{b} + \frac{b}{a}\)?

3. Dissection dilemma: the first two figures can be cut into four congruent pieces as shown. Divide the figure on the right into five congruent pieces.

4. Expand: \((x + y + 1)^2\)

5. Scott cut out a letter (from the English alphabet) and then folded it once. This is what he got:
He informs you that the letter was not an “L”. What letter was it?

6. Write down the next two terms in the following sequence:

\[
\frac{1}{1}, \frac{2}{3}, \frac{5}{8}, \frac{13}{21}, \ldots
\]

7. What is the angle between two adjacent sides of a regular pentagon?

**Stage I**

**Round 3**

1. You have 10 machines, each producing identical steel cubes weighing 50 grams. However, one of the machines is defective and produces steel cubes weighing only 49 grams.

You have a scale that measures weight in grams. Describe how to perform *one* weighing (by putting whatever you like on the scale) to determine which machine is defective.

2. Find the area of the triangle having corners at the points (0, 1), (4, 0), and (2, 3).

3. Find the length of the diagonal:
Spot Prize

What was Gauss’s first name?

Stage II Round 1: Blitz Round

1. A Möbius band is made as follows: take a rectangular strip of paper, then glue the two short edges together with a half twist.

Now the Möbius band is cut along the dotted line down the middle. How many pieces do you get?

2. The two bolts below are held between your thumbs and forefingers and twisted as shown. Will the two heads (a) move toward each other, (b) move apart, or (c) remain the same distance from each other?
3. What percentage of $\frac{5}{4}$ is $\frac{4}{5}$?

4. Compute the sum of all the odd numbers between 1 and 100 (including 1).

5. What is the largest prime number less than 150?

6. In a room of 5 people everyone shakes everyone else’s hand (but not their own). How many handshakes were made in all?

7. What is the maximum number of pieces one can obtain by cutting a pizza with three straight cuts of a cutting wheel?

8. You are given two standard dice, one blue and one red. In how many ways can you roll a 7?

9. Write 17 as a binary number. (For example: $3 = 1 \cdot 2^1 + 1 \cdot 2^0$, and so 3 is 11 in binary.)

10. How many squares are there in a $3 \times 4$ grid?

\[
\begin{array}{cccc}
\hline
\hline
\hline
\hline
\end{array}
\]

<table>
<thead>
<tr>
<th>Stage II</th>
<th>Round 2</th>
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1. Given that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$, find:

(a) $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

(b) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

2. You are hired for a job for 7 days, for which you will be paid a bar of gold. However, you may stop at the end of any day between Day 1 and Day 7. Your payment will be a proportional fraction of the bar (eg. if you stop on Day 3, you get $\frac{3}{7}$ of the bar).
What is the smallest number of pieces that your employer can cut the bar into, before the work begins, and be able to pay you on any day? Where should the cuts be made?

3. (a) Find two whole numbers whose sum is equal to their product.
   (b) Do the same as in part (a) except now the numbers can be fractions. If one of them is 3, what can the other number be?

**SPOT PRIZE**

What is the most prestigious prize in mathematics, and how often is it awarded?

**STAGE III**  
**ROUND 1**

1. The segment $xy$ has length 10. Find the shaded area:

2. Which of the grids below can be tiled by $1 \times 2$ dominoes?

**STAGE III**  
**ROUND 2**

1. The floor of a square room is to be tiled according to the pattern below. Both white sections are square and the larger white square has exactly two tiles more on each side than the smaller one.
If 100 white tiles are needed in all, how many gray tiles are required?

2. How many points with integer coordinates are there inside the square region with corners at \((10\frac{1}{2}, 0), (0, 10\frac{1}{2}), (-10\frac{1}{2}, 0), (0, -10\frac{1}{2})\) ?

**Spot Prize**

Name a famous living mathematician.

**Sudden Death**

1. How many even numbers are there between 1 and 99?

2. How many numbers between 9 and 99 are divisible by 5?

3. How many times does the graph of \(y = x^2 - 1\) meet the \(x\)-axis?

4. What is the remainder when you divide 1987 by 5?
5. What is the angle \( A \)?

6. One of the angles of an isosceles triangle is 60°. What are the other two angles?

7. You toss a coin four times. In how many ways can you get exactly two heads and two tails?

8. Is 47 a prime number?
Stage I, Round 1

#1. No. The region has 21 blocks while any region covered by 1x2 dominoes must have an even number of blocks.

#2. $2^{\sqrt{2}} = 2^{1.4146...} > 2^1 = (\sqrt{2})^2$

So $2^{\sqrt{2}}$ is bigger than $\sqrt{2}^2$.

#3. Take any Pythagorean triple and multiply every number by 2; for example

$6^2 + 8^2 = 10^2$
$10^2 + 24^2 = 26^2$

etc.
Stage I, Round 2 (Blitz)

#1. White - it must be a polar bear; the explorer is at the North Pole.

#2. For \( a, b \) positive real numbers, \( \frac{a}{b} + \frac{b}{a} \geq 2 \).

#3.

#4. \((x + y + 1)^2 = x^2 + y^2 + 1 + 2xy + 2x + 2y\)

#5. The letter "F" (here displayed upside down; fold \( \left[\begin{array}{cc}
1 & 1 \\
1 & 0
\end{array}\right] \))

#6. \(\frac{1}{1}, \frac{2}{3}, \frac{5}{8}, \frac{13}{21}, \frac{34}{55}, \frac{89}{84}, \frac{233}{377}, \ldots\)

(from the Fibonacci sequence)

#7. interior angle of a regular \( n \)-gon = \( \frac{(n-2) \pi}{n} \)

\( \therefore \) interior angle of a regular pentagon

\[ = \frac{3\pi}{5} = 108^\circ \].
Stage I, Round 3

#1. Pick one cube from the first machine, two cubes from the second machine, three cubes from the third ... , ten cubes from the tenth machine.

Total # of cubes = 55

Expected weight = $55 \times 50 = 2750$ gms.

(if none defective)

Actual weight = $2750 - \frac{1}{gm.} \times \# \text{ of defective machine}$

$\therefore \# \text{ of defective machine} = 2750 - \text{actual weight of all 55 cubes.}$

#2.

Area (rectangle) = $4 \times 3 = 12$ units

Area (outer triangles) = $\frac{1}{2} \times 1 \times 4 + \frac{1}{2} \times 2 \times 3 + \frac{1}{2} \times 2 \times 2$

= $\frac{1}{2} \times 14 = 7$ units

$\therefore$ Area (triangle) = $12 - 7 = 5$ units.

#3. Length of the diagonal

= radius of the circle enclosing the box

= $10$. 
Stage II, Round 1: Blitz

#1. A single piece; a doubly twisted band.

#2. Neither bolt moves.

#3. \[
\frac{45}{54} \times 100 = \frac{16}{25} \times 100 = 64\%.
\]

#4. 
\[
1 + 3 + 5 + 7 + \ldots + 99 = \frac{(1+99) + (3+97) + (5+95) + \ldots + (49+51)}{25 \text{ pairs}}
\]
\[
= 25 \times 100 = 2500.
\]

#5. \[
\frac{149}{4} \text{ is prime}.
\]

#6. \[
\# \text{ of handshakes} = \frac{5 \times 4}{2} = 10.
\]

#7. Maximum \# of pieces = 7.

#8. \[
\# \text{ of ways to roll a 7} = 6.
\]
\[
[(1,6), (6,1), (2,5), (5,2), (3,4), (4,3)]
\]

#9. 
\[
17 = 1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0
\]
So \[
17_2 = 10001.
\]

#10. \[
\# \text{ of squares} = 20.
\]
Stage II, Round 2

1. (a) \( \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots = \frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \cdots \)
   \[ = \frac{1}{4} \left( \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right) = \frac{\pi^2}{24} \]

(b) \( \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \left( \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right) - \left( \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots \right) \)
   \[ = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{\pi^2}{8} \]

2. Minimum # of cuts = 2 (3 pieces)
   Cut \( \frac{1}{7} \) and \( \frac{3}{7} \).

3. (a) \( 2 + 2 = 2 \cdot 2 \)
   \( \text{or } 0 + 0 = 0 \cdot 0 \)
   \[ \{ x^2 = 2x \} \]

(b) \( 3 + \frac{3}{2} = 3 \cdot \frac{3}{2} \) (3 + \( x = 3x \))
Stage III, Round 1

#1.

\[
\text{shaded area} = \pi R^2 - \pi r^2 \\
= \pi (R^2 - r^2) \\
= \pi \cdot 5^2 \\
= 25\pi
\]

#2.

cannot be tiled by 1x2 dominoes

(Parity check: more squares of one color than the other.)

can be tiled by 1x2 dominoes.

← for example...
Stage III, Round 2

#1.

Given: \( a = b + 2 \)

\[
\text{\# of white tiles} = a^2 + b^2 = 100
\]

Want:

\[
\text{\# of gray tiles} = 2ab.
\]

\[
a^2 + b^2 = (b+2)^2 + b^2 = 100
\]

\[
\Rightarrow b^2 + 4b + 4 + b^2 = 100
\]

\[
\Rightarrow b^2 + 2b - 48 = 0
\]

\[
\Rightarrow (b+8)(b-6) = 0 \Rightarrow \frac{b=6}{a=8}
\]

\[
\text{\# of gray tiles} = 2ab = 2 \cdot 6 \cdot 8 = 96.
\]

#2.

Use symmetry:

\[
\text{\# of interior points in 1st quadrant} = \text{\# of positive integer solutions to } x+y=k, \ 1 \leq k \leq 9
\]

\[
= 1 + 2 + 3 + \ldots + 9 = 45
\]

\[
\text{\# of integer points on axes (except (0,0)) = 10 each.}
\]

\[
\Rightarrow \text{total \# of integer points} = 4 \cdot 45 + 4 \cdot 10 + 1_{(0,0)} = 221.
\]
#1. Even numbers between 1 and 99 = 49.

#2. How many multiples of 5 between 9 & 99? \( \rightarrow 18 \).

#3. \( y = x^2 - 1 \) meets the X-axis 2 times (at (1,0) and at (-1,0).)

#4. \( 1987 \div 5 \) yields a remainder of 2.

#5. \[
\begin{align*}
\triangle & 40^\circ \\
A & = 40^\circ + 70^\circ = 110^\circ.
\end{align*}
\]

#6. One of the angles of an isosceles \( \triangle \) is 60°. The other two angles must be 60° and 60°.

#7. You toss a coin four times.

\# of ways to get 2 heads & 2 tails = 6.

#8. Yes, 47 is a prime number.