TOPOLOGY QUALIFYING EXAM — JANUARY 2022

1. Definitions and examples

Please clearly state definitions, and describe your examples precisely. You do NOT need to prove that your examples have the required properties. **Solve *all* problems in this section.**

Problem 1.1. Let $(X, \mathcal{T})$ be a topological space, and $A \subset X$ a subset. Define the *subspace topology* on A. Give an example of a set X, with two DIFFERENT topologies $\mathcal{T}, \mathcal{T}'$, and a subset $A \subset X$, such that the two subspace topologies coincide.

Problem 1.2. Let $(X, \mathcal{T})$ be a topological space, $\sim$ an equivalence relation on the set X, and let $X/\sim$ denote the set of all equivalence classes. Define the *quotient topology* on the set $X/\sim$. Give an example of equivalence relation $\sim$ on $\mathbb{R}^2$ (with its standard topology) such that $\mathbb{R}^2/\sim$ is homeomorphic to $[0, \infty)$.

Problem 1.3. Define when two topological spaces are *homotopy equivalent*. Give an example of a pair of topological spaces that are homotopy equivalent, but are NOT homeomorphic.

Problem 1.4. Given a continuous map $f : X \to Y$ between topological spaces, and $x_0 \in X$ a basepoint, define the *induced homomorphism* f_* between the appropriate fundamental groups. Give an example where f is injective, but f_* is NOT injective.

Problem 1.5. Given a continuous map $p : X \to Y$ between topological spaces, define what it means for an open subset $U \subset Y$ to be *evenly covered*. Give an example of $p : X \to Y$, that is NOT a homeomorphism, such that EVERY open subset $U \subset Y$ is evenly covered.

2. Point-set topology

Solve 3 of the following problems. Please clearly indicate what theorems you are using.

(The set $\mathbb{R}$ of real numbers is always endowed with the standard topology, $\mathbb{R}^n$ with the product topology, and subsets like $[0,1] \subset \mathbb{R}$, $S^1 \subset \mathbb{R}^2$, etc with the subspace topology.)

Problem 2.1. Let X be a topological space, $A \subset X$ a subset, and $f : A \to Y$ a continuous map to some other topological space Y. Denote by $\bar{A}$ the closure of A.

(a) Prove that if Y is Hausdorff, then f admits at most one continuous extension $g : \bar{A} \to Y$.
(b) Give an example where f admits NO continuous extension to $\bar{A}$.
(c) Give an example (with Y not Hausdorff) where f admits more than one continuous extension to $\bar{A}$.

Problem 2.2. (a) Let X be a compact Hausdorff topological space. Prove that any continuous surjection $p : X \to X$ is a quotient map.
(b) Let $X \subset \mathbb{R}^2$ be the union of $\{(x,y) \mid xy = 1\}$ with the origin $\{(0,0)\}$, and let $p : X \to \mathbb{R}$ be given by $p(x,y) = x$. Prove or disprove: "p is a quotient map".

Problem 2.3. Let (X,d) be a metric space, and $A \subset X$ a subset. Let $f : X \to \mathbb{R}$ be given by $f(x) = \inf_{a \in A} d(x,a)$.

(a) Prove that f is continuous. (Hint: Use the triangle inequality)
(b) Show that the closure $\bar{A}$ of A is equal to $\{x \in X \mid f(x) = 0\}$.

Problem 2.4. (a) Let $U \subset \mathbb{R}^n$ be an OPEN subset. Prove that if U is connected, then U is path-connected.
(b) Let X be a topological space, and $U \subset X$ a subset. Prove directly from the definition that, if U is connected, then its closure $\bar{U}$ is also connected.

Problem 2.5. Think of $\mathbb{R}^{n^2}$ (for $n \geq 2$) as the space of all $n \times n$ matrices with real entries, by indentifying a matrix (a_{ij}) with the vector

$$(a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, a_{22}, \ldots, a_{nn}).$$

(a) Let $X \subset \mathbb{R}^{n^2}$ be the set of orthogonal matrices, i.e.,

$$X = \{A \in \mathbb{R}^{n^2} \mid A.A^t = \mathrm{Id}\}.$$

Prove that X is compact. (Here A^t denotes the transpose of A, and Id the $n \times n$ identity matrix.)

(b) Let $Y = \{A \in \mathbb{R}^{n^2} \mid \det(A) = 1\}$. Prove that Y is NOT compact.

3. Fundamental group and covering spaces

Solve 3 of the following problems. Please clearly indicate what theorems you are using.

(The set $\mathbb{R}$ of real numbers is always endowed with the standard topology, $\mathbb{R}^n$ with the product topology, and subsets like $[0, 1] \subset \mathbb{R}$, $S^1, D^2 \subset \mathbb{R}^2$, etc with the subspace topology.)

Problem 3.1. Let $S^1 \subset D^2 \subset \mathbb{R}^2$ denote the unit circle and the unit disk, respectively. Let $D^2 \vee D^2$ be obtained by gluing a basepoint $x_0 \in S^1$, so that $S^1 \vee S^1$ is a subset of $D^2 \vee D^2$. Prove that there is no retraction $r : D^2 \vee D^2 \to S^1 \vee S^1$.

Problem 3.2. Suppose the topological space X is path-connected, locally path-connected, and semilocally simply-connected, and let $\tilde{X}$ be its universal cover. Assume $\pi_1(X)$ is infinite. Prove that $\tilde{X}$ is NOT compact.

Problem 3.3. Let $X \subset \mathbb{R}^2$ be the union of the unit circle S^1 (centered at the origin), its vertical diameter, and its horizontal diameter, that is:

$$X = S^1 \cup (\{0\} \times [-1, 1]) \cup ([-1, 1] \times \{0\}) .$$

(With the subspace topology induced from $\mathbb{R}^2$.) Use the van Kampen Theorem to compute the fundamental group of X.

Problem 3.4. Recall that projective n-space is defined as the quotient space $\mathbb{R}P^n = S^n/\sim$ where $v \sim -v$ for all $v \in S^n$. Let $X = \mathbb{R}P^2 \times \mathbb{R}P^3$, and fix a basepoint $x_0 \in X$.

(a) Compute the fundamental group $\pi_1(X, x_0)$.

(b) How many covering spaces $(Y, y_0) \to (X, x_0)$ (with Y connected) are there, up to basepoint-preserving isomorphism?

(c) How many of these convering spaces are regular (i.e. normal)? Why?

Problem 3.5. Let $f : S^1 \to S^1$ be a null-homotopic map. Prove that f has a fixed point.