Algebra Qualifying Exam — August 2021

Instructions

- Do 3 out of the 4 problems in Part I and 3 out of the 4 problems in Part II.
- Give full and clear justification of your solutions.
- Throughout, p denotes a prime number and $\mathbb{F}_p$ denotes a field with p elements.
- For a ring R, $Z(R)$ denotes the center of R and $M_n(R)$ denotes the ring of $n \times n$ matrices over R. If R is commutative, $\mathrm{GL}_n(R) = M_n(R)^\times$.

Part I

1. Fix a prime p. Consider the $\mathbb{F}_p$-vector space $V = \mathbb{F}_p^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{F}_p \right\}$ and subspace $W = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \mathbb{F}_p \right\}$. Let $\mathrm{GL}_2(\mathbb{F}_p)$ act on V in the usual way by matrix multiplication. Put

$$G = \{ g \in \mathrm{GL}_2(\mathbb{F}_p) : gw \in W \text{ for all } w \in W \}.$$

   (a) Show G is a subgroup of $\mathrm{GL}_2(\mathbb{F}_p)$.

   (b) Determine the order of G.

   (c) For which p is G abelian?

2. Determine all pairs of prime numbers (p, q) which satisfy the following: every group G of order pq with a normal subgroup of order p is abelian.

3. Describe (with proof), as explicitly as you can, both the set of prime ideals and the set of maximal ideals for each of the following rings:

   (a) $\mathbb{Z}/6\mathbb{Z}$

   (b) $\mathbb{Q}[x]$

   (c) $\mathbb{Z}[x]/(x^2 - x - 1)$

4. Let R be a ring, and M be a (left) R-module. Recall M is *simple* (or *irreducible*) if $M \neq 0$ and the only nonzero submodule of M is M itself.

   (a) Prove that M is cyclic if M is simple.

   (b) Take $R = \mathbb{Z} \times \mathbb{Z}$. Exhibit a nonzero cyclic R-module which is not simple.

   (c) Again with $R = \mathbb{Z} \times \mathbb{Z}$, classify, as explicitly as you can, all simple R-modules up to isomorphism.

Part II

5. Let $f(x) = x^4 + 3 \in \mathbb{Q}[x]$.
   (a) Determine a splitting field K of $f(x)$, and give its degree over $\mathbb{Q}$.
   (b) Determine all subfields of $K/\mathbb{Q}$, and give their degrees over $\mathbb{Q}$.
   (c) Determine the structure of $\mathrm{Gal}(K/\mathbb{Q})$ (as an abstract group you know).

6. Let F be a field. Let Σ be the set of irreducible polynomials $f(x) \in F[x]$ such that a splitting field K of $f(x)$ has degree 6 over F. Determine $\{\deg f(x) : f(x) \in \Sigma\}$ in the following two cases:
   (a) $F = \mathbb{Q}$;
   (b) F is a finite field.

7. (a) Show that every quadratic field extension $F/\mathbb{Q}$ embeds as a subring of $M_2(\mathbb{Q})$.
   (b) Fix two quadratic fields F and K, as well as an embedding $i : F \hookrightarrow M_2(\mathbb{Q})$. Determine the set of embeddings $j : K \hookrightarrow M_2(\mathbb{Q})$ such that $i(x)$ and $j(y)$ commute for every $x \in F$, $y \in K$.

8. (a) Compute a complete set of representatives of the equivalence classes of quadratic forms on the 3-dimensional vector space $V = \mathbb{F}_5^3$ over $\mathbb{F}_5$.
   (b) Which of these forms are isotropic?
   (c) Which of these forms are universal?