Qualifying Exam in Topology, Fall 2020

1. Concepts

Answer all the questions in this section.

1. Prove or give a counterexample: For all topological spaces X and subsets $A \subseteq X$, $\overline{\operatorname{int}(A)} = \overline{A}$.

2. Prove or give a counterexample: Homeomorphic metric spaces are isometric.

3. Let $\mathbb{Z}$ have the topology whose subbasis consists of the sets $\{U_m\}$ where $U_m = \mathbb{Z} \setminus \{m\}$. I.e. the complement of any point is in the subbasis. Is $\mathbb{Z}$ with this topology connected? Is it compact? Justify your answer.

4. Suppose X, Y, Z are all path-connected spaces (but not necessarily semilocally simply connected). Suppose $f : X \to Y$ and $g : Y \to Z$ are both covering maps of degree 2. Is $g \circ f$ necessarily a covering map of finite degree? Prove or give a counterexample.

5. Suppose $\gamma \in \pi_1(X, x)$ and $\gamma' \in \pi_1(X, x)$ are conjugate. Suppose $a : S^1 \to X$ and $b : S^1 \to X$ are loops representing the homotopy classes γ, γ' respectively. What can you say about the *unbased* homotopy classes of a and b?

2. Point Set Topology (Mostly)

Solve 3 of the following problems. In your answers, indicate what theorems that you are using. If you are using a theorem you found in the book that you are using for the exam, give the theorem number or the section it's in plus the page number. If you are using a theorem/result that you remember from class, then describe what the theorem/result says.

(1) Let the set $\{0, 1\}$ have the discrete topology. Prove that $X = \{0, 1\}^{\mathbb{N}}$ (with the product topology) is compact, Hausdorff, and totally disconnected. Deduce that X does not have the discrete topology.

(2) (a) Prove that a bijective continuous function $f : S^2 \to S^2$ is a homeomorphism.
(b) Suppose $f : S^2 \to \mathbb{R}^2$ is continuous and the image $f(S^2)$ is not a single point. Prove that $f(S^2)$ is uncountable and that no point of $f(S^2)$ is isolated.

(3) Suppose $X = \bigcup_{i \in \mathbb{Z}} A_i$, that A_n is connected for all n and $A_n \cap A_{n+1} \neq \emptyset$ for all $n \in \mathbb{Z}$. Prove that X is connected.

(4) Prove that there is a quotient map $S^1 \times [-1, 1] \to S^2$. (I want to see a detailed argument here.)

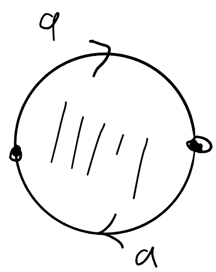
(5) Let X be a compact Hausdorff space. Prove X is metrizable if and only if X is 2nd countable.

3. Fundamental Group and Covering Spaces (Mostly)

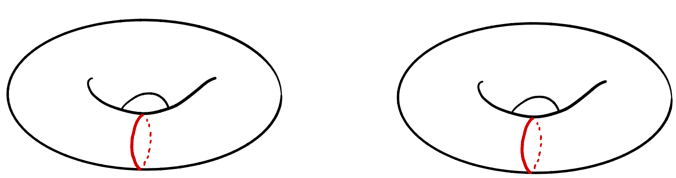
Solve 3 of the following problems. In your answers, indicate what theorems that you are using. If you are using a theorem you found in the book that you are using for the exam, give the theorem number or the section it's in plus the page number. If you are using a theorem/result that you remember from class, then describe what the theorem/result says.

(1) Describe all connected index 4 covers of $\mathbb{R}P^2 \vee \mathbb{R}P^2$. Indicate which index 4 covers are regular and describe their deck transformations.

(2) Show that there is no retraction $r : \mathbb{R}P^2 \to S^1$ where S^1 is the loop a indicated in the picture of $\mathbb{R}P^2$.

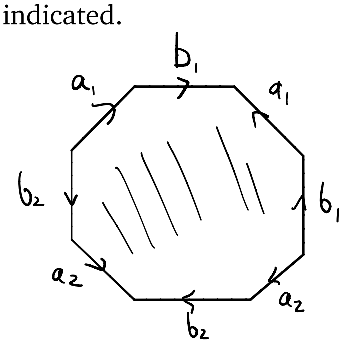


(3) Let X be the space obtained by gluing two tori along circles in each torus. I.e. let $T = S^1 \times S^1 \times \{0\}$ and $T' = S^1 \times S^2 \times \{1\}$ (with the usual topologies), and then X is the quotient space $T \sqcup T'/\sim$ where $(s, t, 0) \sim (s, t, 1)$ if and only if $t = 1 = e^0 \in S^1 \subset \mathbb{C}$. (See the figure below.) Use Van Kampen to compute $\pi_1(X, x)$.



The tori are glued along the red curves.

(4) Prove that the closed surface of genus 2 does not cover the 2-dimensional torus. Recall that a closed surface of genus 2 is homeomorphic to the following quotient of an octagon with sides identified as indicated.



(5) Construct a compact CW-complex X whose fundamental group is $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. (You must prove your space has this fundamental group.) Is the universal cover of X compact or non-compact? Prove your answer.