Algebra Qualifying Exam – August 2020

- *Please try to explain your work clearly and write neatly.*
- *Full credit for complete answers to any n questions where n is the unique positive integer such that the symmetric group S_n admits an outer automorphism.* [1] *If you do more than n questions, we'll take your best n.*
- *By convention, all rings and subrings have (multiplicative) identity elements. For S a ring, $S^\times$ denotes the group of units (or unit group) of S. For q a prime power, we write $\mathbb{F}_q$ for the field with q elements.*
- *Good luck!*

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1. For each statement, provide a proof if true, a counterexample if false.

   a. Every group of order 35 is cyclic.

   b. Every group of order 180 is solvable.

   c. If there is a non-trivial homomorphism from $\mathbb{Q}$, the additive group of rational numbers, to a group A then A is infinite.

2. Let $p > 2$ be a prime and set

$$A = \{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a, b \in \mathbb{F}_p, a \neq 0 \}.$$

   Then A is a subgroup of $\mathrm{GL}_2(\mathbb{F}_p)$, the multiplicative group of invertible 2×2 matrices with entries in $\mathbb{F}_p$.

   a. Prove that A has a unique Sylow p-subgroup.

   b. Let l be a prime divisor of $p - 1$. How many Sylow l-subgroups does A have?

   c. Show that A is not nilpotent.

3. a. Explain why the subring $\mathbb{Z}[\sqrt{2}, e, \pi]$ of $\mathbb{R}$ generated by $\sqrt{2}$, e and π is Noetherian.

   b. Let $\mathcal{R}$ be a subring of $\mathbb{R}$ such that the polynomial ring $\mathcal{R}[\mathrm{X}]$ is Noetherian. Show that $\mathcal{R}$ is Noetherian.

4. Let R be a commutative ring and write $\mathfrak{n}$ for the set of nilpotent elements in R. (Recall that $r \in R$ is *nilpotent* if $r^n = 0$ for some positive integer n.)

   a. Show that $\mathfrak{n}$ is an ideal in R.

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[1] By work of Hölder, $n = 6$.

b. Prove that $1 + \mathfrak{n} \subset R^\times$ where $1 + \mathfrak{n} = \{1 + x : x \in \mathfrak{n}\}$. (Hint: consider the identity $1 - \mathrm{X}^n = (1 - \mathrm{X})(1 + \mathrm{X} + \cdots + \mathrm{X}^{n-1})$ in $\mathbb{Z}[\mathrm{X}]$.)

c. Show that the quotient map $x \mapsto x + \mathfrak{n} : R \to R/\mathfrak{n}$ induces an isomorphism of groups $R^\times / 1 + \mathfrak{n} \simeq (R/\mathfrak{n})^\times$.

5. Let p and l be primes. Assume that l divides $p - 1$ and that $\mathrm{X}^l - a \in \mathbb{F}_p[\mathrm{X}]$ has no root in $\mathbb{F}_p$. Prove that $\mathrm{X}^l - a$ is irreducible over $\mathbb{F}_p$.

6. Let K/F be a Galois extension of fields with $\mathrm{Gal}(K/F) \simeq A_4$, the alternating group on a set with four elements.

a. Show that there is no quadratic intermediate field. That is, there is no field K' with $F \subset K' \subset K$ and $[K' : F] = 2$.

b. Show that there is a unique field K' with $F \subsetneq K' \subsetneq K$ and K'/F Galois.

7. Consider the ring $R = \mathbb{Q}[\mathrm{X}] \,/\, \big(\mathrm{X}^4 - 1\big)$.

a. How many maximal ideals does R have?

b. Write ${}_4R^\times$ for the subgroup of $R^\times$ consisting of elements $r \in R$ such that $r^4 = 1$. Express ${}_4R^\times$ as a product of cyclic groups.

8. a. Let R be a commutative ring and let I and J be ideals in R such that R/I and R/J are isomorphic as R-modules. Prove that $I = J$.

b. Let $\mathrm{M}_2(\mathbb{F}_q)$ denote the ring of 2×2 matrices with entries in the finite field $\mathbb{F}_q$ (with q elements). Determine the number the elements $A \in \mathrm{M}_2(\mathbb{F}_q)$ such that $A^2 = 0$. Your answer should be a simple expression in terms of q.