Algebra PhD Qualifying Examination – August 2017

Instructions:

- Please write a neat, clear, thoughtful, and hopefully correct solution to each of the following problems. Please show *all* relevant work.
- You should do as many problems as the time allows. You are not expected to answer all parts of all questions in order to pass the exam.
- Each problem is worth the same. Partial credit will be given, but a complete solution of one problem is worth more than partial work on two problems.
- Good luck.

Problems:

1. Let G be a group and H a subgroup of G of order $n < \infty$ and assume that H is the unique subgroup of G of that order. Prove H is normal.
2. (a) How many isomorphism classes of abelian groups of order 56 are there? Give a representative for each isomorphism class.
   (b) Prove that if G is a not-neccesarily-abelian group of order 56, then either the Sylow-2 subgroup or the Sylow-7 subgroup is normal.
   (c) Give two non-isomorphic groups of order 56 where the Sylow-7 is normal, and the Sylow-2 subgroup is not normal. Be sure to prove that your two groups are not isomorphic.
3. Let I be an ideal of a commutative ring with identity, R. We call I Stoopian if it has the property that whenever there is $x, y \in R$ with $xRy \subseteq I$ it implies either $x \in I$ or $y \in I$. Prove that I is Stoopian if and only if I is a prime ideal.
4. Let R be a commutative ring with identity. Suppose that M is a free R-module with a finite basis X. Let I be a proper ideal of R.
   (a) Prove that IM is a submodule of M.
   (b) Prove that M/IM is a free R/I-module with basis X_0, where X_0 is the image of X under the canonical map $M \to M/IM$.
5. Let F be a field.
   (a) Let V be an n-dimensional F-vector space and let W be an m-dimensional F-vector space. Please define the vector space $V \otimes_F W$, give its dimension, and please give a basis.
   (b) Let $R = F[x, y]$ and $S = F[y]$. Fix $\lambda \in F$. Let $F_\lambda = F$ as a vector space and view it as a left S-module with the action given by $p(y).v = p(\lambda)v$. Let R be a right S-module by right multiplication.
      i. Prove that $R \otimes_S F_\lambda \cong F[x, y]/(y - \lambda)$ as vector spaces.
      ii. Prove that since R is actually an (R, S)-bimodule, $R \otimes_S F_\lambda$ is a left R-module. Prove that the above isomorphism is actually as left R-modules.
   (c) Please explicitly calculate the abelian group $\mathbb{Z}_{12} \otimes_{\mathbb{Z}} \mathbb{Z}_4$.
6. (a) Let K denote the splitting field of $x^3 - 5$ over $\mathbb{Q}$.
   (b) Determine K.
   (c) Determine the Galois group $G = \text{Gal}(K/\mathbb{Q})$.
   (d) Determine all intermediate fields between the splitting field K and $\mathbb{Q}$.
7. Let K/F be a Galois extension of fields such that $G = \text{Gal}(K/F)$ is a finite group. Prove that if there is an intermediate field E between K and F which is a degree 2 extension of F, then it must be Galois.