Jan 13, 2017 — TOPOLOGY QUALIFY EXAM — 2:00-5:00pm

Instructions: You have three hours to finish the exam. There are three parts to the exam. For the proofs, provide justifications and make your arguments clear, but try to avoid excess detail.

I

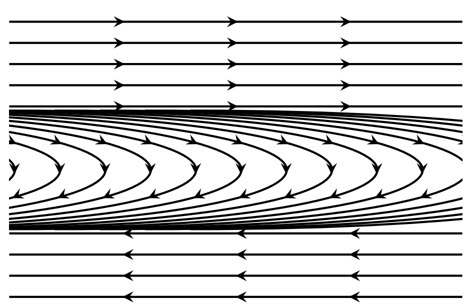
Definitions and statements of theorems (solve ALL problems)

1. Let $\{X_\alpha \mid \alpha \in J\}$ be an non-empty indexed family of non-empty topological spaces. Define the product topology on $X = \prod_{\alpha \in J} X_\alpha$.
2. Define what it means for a topological space to be compact and locally-compact.
3. State the Tychnoff theorem.
4. Define what it means for a subspace $A \subset X$ to be a deformation retract of X.
5. Define what it means for a space Y to be a covering space of X.

II

Solve FOUR of these problems.

1. Which of the following are topological manifolds? Explain.
   (a) n-sphere
   (b) A figure eight
   (c) The closure of the topologists' sine curve
2. Let Y be a Hausdorff space and consider continuous functions $f, g : X \to Y$. Show that the set $\{x \in X \mid f(x) = g(x)\}$ is closed in X.
3. Is $\mathbb{Q}$ locally compact? Explain.
4. Give an example of a space that is path connected by not locally connected.
5. Consider an action of $\mathbb{R}$ on the plane whose flow lines are horizontal lines above $y = 1$ and below $y = -1$ and U-shaped in between, see picture. Say two points are equivalent if they lie on the same flow line. Is the quotient space Hausdorff? Give an explicit description of the quotient space.



III

Solve FOUR of these problems.

1. Suppose $g\colon S^2 \to S^2$ is a continuous map from the 2-sphere to itself such that $g(-x) \neq g(x)$ for all $x \in S^2$. Show that g must be surjective.

2. Prove that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^3$. Here each space has the standard (euclidean) topology. State any results in class that you use in your proof.

3. Show that if a path-connected, locally path-connected space X has finite fundamental group, then every map $X \to S^1$ is nullhomotopic.

4. The suspension of a topological space X is the quotient space $\Sigma X = X \times [0,1]/\sim$ where $(x,t) \sim (y,s)$ if and only if either $(x,t) = (y,s)$ or $s = t = 1$ or $s = t = 0$. Prove that the suspension ΣX is simply connected if X is path-connected.

5. Let X be the wedge of the torus with the circle: $X = (S^1 \times S^1) \vee S^1$. Use Van Kampen's theorem to write a presentation for the fundamental group of X. Draw all connected 3-sheeted covering spaces of X up to homeomorphism.