Provide justification for all of your answers.

1. (10 points) Let G be a group and let $\mathcal{Z}(G)$ be its center. Prove
   (a) If $G/\mathcal{Z}(G)$ is cyclic, then G is abelian.
   (b) If G is of order p^2, where p is prime, then G is abelian.

2. (10 points) A group G is solvable if it has a series
$$\{1\} = G_0 \lhd G_1 \lhd \ldots \lhd G_k = G$$
such that G_{i-1} is normal in G_i and the quotient group G_i/G_{i-1} is abelian for $1 \leq i \leq k$. Prove that a group G of order $105 = 3 \cdot 5 \cdot 7$ is solvable.

3. (10 points) Let R be a commutative ring with 1. The *Jacobson radical* of R, denoted $J(R)$ is the intersection of all maximal ideals of R.
   (a) Prove that $a \in J(R)$ if and only if $1 + ar$ is a unit in R for all $r \in R$.
   (b) Let k be a field. Prove $J(k[x]) = 0$.

4. (20 points)
   (a) Is $\langle x^2 + 1, 3\rangle$ a prime ideal of $\mathbb{Z}[x]$? Prove your answer is correct.
   (b) Prove or disprove: If p is prime, then every non-zero prime ideal of $\mathbb{Z}/p\mathbb{Z}[x]$ is maximal.
   (c) Prove or disprove: Any irreducible element in an integral domain is prime.
   (d) Prove or disprove: Any prime element in an integral domain is irreducible.

5. (10 points) Let $F \subset E \subset K$ be fields.
   (a) Prove that K is algebraic over F if and only if K is algebraic over E and E is algebraic over F.
   (b) Give an example where K/E and E/F are finite Galois extensions, but K/F is not Galois.

6. (10 points) Suppose $K = F(\alpha)$, $\alpha \notin F$, is a Galois extension of F. Assume that there exists an element $\sigma \in Gal(K/F)$ satisfying $\sigma(\alpha) = \alpha^{-1}$. Show that the degree of the extension $[K : F]$ is even and that $[F(\alpha + \alpha^{-1}) : F] = \frac{1}{2}[K : F]$.

7. (15 points) Let $F = \mathbb{F}_{81}$ be the field of 81 elements.
   (a) Find all subfields of F.
   (b) Determine the number of elements $\alpha \in F$ such that $F = \mathbb{F}_3(\alpha)$, where $\mathbb{F}_3$ is the field of 3 elements.
   (c) Find the number of generators for the multiplicative group $F^\times$ of F (i.e. $\beta \in F$ such that $\langle\beta\rangle = F^\times$).