Algebra Qualifying Exam (August 2013)

Solve as many of the eight problems as you can. It is not necessary to solve everything in order to pass the exam.

1) Let p be a prime, and q a positive power of p. Let $\mathbb{F}_q$ be the field with q elements. For a positive integer n, let $G = \mathrm{GL}(n, \mathbb{F}_q)$, and let B be the subgroup of G consisting of upper triangular matrices with 1's on the diagonal. Show that B is a Sylow p-subgroup of G.

2) Let E be the splitting field of $X^5 - 2$ over $\mathbb{Q}$. Prove:

   a) The degree $[E : \mathbb{Q}]$ equals 20.

   b) There exists exactly one intermediate field F with $[E : F] = 5$. The extension $F/\mathbb{Q}$ is normal.

3) For each positive integer n, let ζ_n be a fixed primitive n-th root of unity inside the complex numbers $\mathbb{C}$. For fixed m and n, let d be their greatest common divisor, and v their least common multiple.

   a) Show that $\langle \zeta_m \rangle \cap \langle \zeta_n \rangle = \langle \zeta_d \rangle$. (Here, $\langle g \rangle$ denotes the subgroup of $\mathbb{C}^\times$ generated by an element $g \in \mathbb{C}^\times$.)

   b) Show that there is an exact sequence

   $$1 \longrightarrow \langle \zeta_d \rangle \longrightarrow \langle \zeta_m \rangle \times \langle \zeta_n \rangle \longrightarrow \langle \zeta_v \rangle \longrightarrow 1.$$

   c) Show that $\mathbb{Q}(\zeta_m, \zeta_n) = \mathbb{Q}(\zeta_v)$.

4) Prove the *Translation Theorem of Galois Theory*: Let E/K be a finite Galois extension. Let K'/K be any field extension. Then the extension EK'/K' is also Galois, and its Galois group $G(EK'/K')$ is naturally isomorphic to $G(E/E \cap K')$. (All fields are assumed to be contained in some big field L.)

5) Let R be the ring $\mathbb{Z}[i]$.

   a) Show that $3R$ is a prime ideal in R, but $5R$ is not.

   b) Show that, in fact, $R/3R$ is a field. Which one?

6) Let U, V, W be vector spaces over a field K. Using the universal property of the tensor product, prove that there is a natural isomorphism

   $$U \otimes (V \oplus W) \cong (U \otimes V) \oplus (U \otimes W).$$

7) Let H be a proper, normal subgroup of the symmetric group S_n. Assume that H contains a 3-cycle. Show that H is the alternating group A_n.

8) Prove that there is no integral domain with exactly 10 elements.