**Qualifying Examination** **Real Analysis** **August 2012**

**Name** **Score:**

---

The symbols and notions are as used in the book Real Analysis by Royden 3rd ed.. If the measure is not explicitly specified measurable, integrable etc. refers to the Lebesgue measure on $\mathbb{R}$ or $\mathbb{R}^n$, respectively. You may use either the book Real Analysis by Royden 3rd ed. or Real Analysis by Fitzpatrick-Royden as reference.

Always provide a proof for your answers.

Work as many problems as you can, each problems counts for 20 pts.

The allotted time is three hours.

---

1) a) Let $E$ be an open and dense set in $\mathbb{R}$. Show that for all pairs of real numbers $a < b$ we have

$$\int_{E\cap(a,b)} dx > 0\,.$$

b) For each $r > 0$ find an open and dense subset $E_r$ of $\mathbb{R}$ such that

$$\int_{E_r} dx \leq r\,.$$

2) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ given by

$$f_\alpha(x) = \begin{cases} \dfrac{1}{|x|^\alpha}\,, & \text{if } x \in \mathbb{R}^n \setminus \mathbb{Q}^n \\ 0\,, & \text{otherwise.} \end{cases}$$

a) Show that $f_\alpha$ is measurable with respect to the Lebesgue measure on $\mathbb{R}^n$, for all $\alpha \in \mathbb{R}$,

b) Find all $\alpha$ such that $f_\alpha$ is in $L^1(D)$ in case
  i) $D = \mathbb{R}^n$.
  ii) $D = \mathbb{R}^n \setminus B(0,1)$.
  iii) $D = B(0,1)$.

Here $B(x,r)$ is the ball centered at $x$ with radius $r$.

3) Let $f$ be a measurable function on an bounded interval $D = [a,b]$. Show that for every $\epsilon > 0$ there is a continuous function $g$ such that

$$m(\{x \in D \mid f(x) \neq g(x)\}) < \epsilon\,.$$

4) Let $[a,b]$ a bounded closed interval and let $f$ and $g$ be two absolutely continuous functions on $[a,b]$. Prove that the product $fg$ is absolutely continuous.

5) Let $X$ be a countable set and $c$ the counting measure on $X$. Show that the product measure $\mu = c \times c$ is the counting measure on $X \times X$.

6) Let $(X, \mathcal{A}, \mu)$ be a measure space and let $g$ be an integrable function such that

$$|\int_X fg d\mu| \leq M \int |f| d\mu \,,$$

for all integrable functions $f$.
Show that $\| g \|_\infty \leq M$.

7) Let $f$ the cumulative distribution function of a Borel measure $\nu$ on the Borel sets of $\mathbb{R}$ be given by

$$f(x) = \begin{cases} 0, & \text{if } x < 0\,, \\ \sqrt{x+1}, & \text{if } x \geq 0\,, \end{cases}$$

that is $\nu(-\infty, x) = f(x)$.

a) If $\lambda$ is the (restriction of) the Lebesgue measure on the Borel sets, find the Lebesgue Decomposition of $\nu$ with respect to $\lambda$.

b) What is the Randon-Nykodym derivative of the absolutely continuous part of $\nu$ with respect to $\lambda$.

8) Let $(X, \mathcal{B}, \mu)$ be a measure space. Let $f$ be integrable function and $f_n$ a sequence of integrable functions such that $\int_X |f_n - f| dx \to 0$ as $n \to \infty$

a) Show that there is a subsequence $f_{n_k}$ of $f_n$ converging to $f$ almost everywhere.

b) Is, in general, the same true for the sequence itself? If not, provide a counterexample.

9) For $(x, y)$ in $(-\pi, \pi) \times \mathbb{R}$ consider the function

$$f(x,y) = \begin{cases} (\sin x)(\frac{1}{|y|})\,, & \text{if } y \neq 0 \\ 0\,, & \text{otherwise} \end{cases}$$

a) Show that $g(y) = \int_{(-\pi,\pi)} f(x,y)dx$ is integrable over $\mathbb{R}$.

b) In this case, do we have $\int_{\mathbb{R}} (\int_{(-\pi,\pi)} f(x,y)dx)dy = \int_{(-\pi,\pi)} (\int_{\mathbb{R}} f(x,y)dy)dx$ ?
If not point out why neither the theorem of Fubini nor the theorem of Tonelli is applicable here.

10) Prove that $L^\infty(\mathbb{R})$ is a Banach Space.