Algebra PhD Qualifying Examination – August 2012

1. Let G be a finite group and let H be a subgroup of G such that $[G : H] = 2$. Suppose K is a subgroup of G of odd order. Show that K is contained in H.

2. Let G be a finite group such that $|G| = p^2q$, where p, q are primes such that $p < q$. Show that either G has a normal subgroup of order q or $|G| = 12$.

3. Let $R = \mathbb{Z}[\sqrt{-m}] = \{x + y\sqrt{-m} : x, y \in \mathbb{Z}\}$, where m is an odd, square-free integer such that $m \geq 3$.

   (a) Find all the units in R.

   (b) Show that 2 and $1 + \sqrt{-m}$ are irreducible in R.

   (c) Show that R is **not** a Unique Factorization Domain.

   (It might be useful to use the norm function on R)

4. Let R be a commutative ring with unity 1. Let J be the intersection of all maximal ideals of R. Show that $1 + J := \{1 + x : x \in J\}$ is a subgroup of the group of units of R. (You may use the fact that every non-trivial ideal of R is contained in a maximal ideal)

5. Let $R = \mathbb{Z}$ and let
$$S = \{3^r \mid r \in \mathbb{Z}_{\geq 0}\}.$$

   (a) Show that S is a multiplicatively closed subset of R.

   (b) Define $S^{-1}R$, including the binary operations which make it into a ring.

   (c) Is $S^{-1}R$ a field? Please prove or disprove.

6. Let K be the splitting field of $x^4 - x^2 - 3$ over $\mathbb{Q}$.

   (a) Determine the Galois group of K over $\mathbb{Q}$.

   (b) Determine the subgroup lattice of the Galois Group.

   (c) Determine the lattice of intermediate fields between $\mathbb{Q}$ and K.

7. Let R be a ring with unity 1 and let M be a unital left R-module. We will write $r.m$ for the action of the element $r \in R$ on the element $m \in M$. Let A and B be submodules of M.

   (a) Prove that
$$A + B := \{a + b \mid a \in A, b \in B\}$$
   is a submodule of M.

   (b) Prove that
$$A \times B := \{(a, b) \mid a \in A, b \in B\}$$
   is a unital left R-module via the action given by
$$r.(a, b) = (r.a, r.b)$$
   for all $r \in R$, all $a \in A$, and all $b \in B$.

   (c) Prove that the map
$$\varphi : A \times B \to A + B$$
   given by $\varphi(a, b) = a + b$ is an R-module isomorphism if and only if $A \cap B = \{0\}$.

8. (a) Let
$$A = \mathbb{Z}/3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}.$$
   As abelian groups, A is isomorphic to which group? Please prove.

(b) Let

$$B = \mathbb{Z}/3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z}.$$

As abelian groups, B is isomorphic to which group? Please prove.

(c) If m, n are positive integers and $d = \gcd(m, n)$, then prove that as abelian groups we have

$$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}.$$

(It may be useful to prove that $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ is a cyclic group generated by the element $\bar{1} \otimes \bar{1}$ and that the order of this element must divide d.)