Algebra PhD Qualifying Examination - August 2011

Please write neat and clear solutions to the following problems, showing all relevant work. Note that all rings are associative with 1 *and that all modules are unital.*

1. Prove or disprove:
   a) For a group G and subgroups H and K, if $[G : H]$ and $[G : K]$ are finite then $[G : H \cap K]$ is also finite.
   b) If N is a normal subgroup of G with N and G/N cyclic then G is a nilpotent group.
   c) Any group of order $5p^k$ (p prime) is solvable.

2. Prove:
   a) The symmetric group S_n is generated by the transpositions $(1\,2), (1\,3), \ldots, (1\,n)$.
   b) The alternating group A_n si generated by the 3-cycles $(1\,2\,3), (1\,2\,4), \ldots, (1\,2\,n)$.

3. Show that there is no simple group of order 120.

4. Prove that any group of order 175 is abelian. Find all groups of order 175 up to isomorphism.

5. Prove or disprove:
   a) Every ring contains a maximal ideal.
   b) In a commutative ring, if a divides b and b divides a then $a = ub$ for some unit u.
   c) The left $M_n(D)$-module D^n of $n \times 1$ column vectors is simple when D is a division ring.

6. Show that if M is a finitely generated module over a Noetherian ring and $\phi : M \to M$ is an epimorphism, then ϕ is an automorphism.

7. Prove that if I and J are ideals of a commutative ring R with $I + J = R$ then R/IJ is isomorphic to $R/I \times R/J$.

8. Describe all $q = p^n$ (in terms of congruence conditions on p and n) for which $x^2 + 1$ is irreducible over the finite field with q elements.

9. Prove:
   a) If G is a finite abelian noncyclic group then there is a positive integer $k < |G|$ such that $g^k = 1$ for all $g \in G$.
   b) A nonzero polynomial f in $F[X]$ has at most $\deg(f)$ roots. (F is a field.)

c) Any finite multiplicative subgroup of a field is cyclic.

10. If E is the splitting field over $\mathbb{Q}$ of $x^p + 2$ in $\mathbb{C}$ (p a prime) show:
    a) $E = \mathbb{Q}(e^{2\pi i/p}, -2^{1/p})$.
    b) $\mathrm{Gal}(E/\mathbb{Q}(e^{2\pi i/p}))$ and $\mathrm{Gal}(E/\mathbb{Q}(-2^{1/p}))$ are cyclic.
    c) $\mathrm{Gal}(E/\mathbb{Q})$ is isomorphic to the product of the cyclic groups in b).