Topology Qualifying Review Exam

May 17, 2010

Name:

- Prove all the statements unless specified otherwise.

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1. Write definitions. Do as the example.

Example: **quotient topology**

Answer: Let X be a topological space, Y be a set, and $q : X \to Y$ be a map. The strongest topology $\mathcal{T}$ on Y that makes q continuous is called the quotient topology on Y for the map q. In fact, $\mathcal{T}$ is constructed as follows: For $V \subset Y$, $V \in \mathcal{T}$ if and only if $q^{-1}(V)$ is open in X.

(1) product topology on $\Pi_{\alpha\in\Lambda}X_\alpha$
(2) locally finite family
(3) Stone-Čech compactification
(4) homotopic realative to a subset
(5) deformation retract

2. Let $\mathbb{R}_\ell$ denote $\mathbb{R}$ with the lower-limit topology (i.e., $\{[a,b) : \ a,b \in \mathbb{R}\}$ is a subbasis). Prove/disprove

2A. $[0,1] \subset \mathbb{R}_\ell$ is compact.

2B. $\mathbb{R}_\ell$ is metrizable.

3. 3A. Let $p : X \to Y$ be a quotient map. Let $f : X \to Z$ be a map that is constant on each set $p^{-1}(\{y\})$ for each $y \in Y$ (so that f induces a map $\overline{f} : Y \to Z$ such that $\overline{f} \circ p = f$). Prove that if f is continuous, then $\overline{f}$ is continuous.

3B. Let $f : S^1(= \partial B^2) \longrightarrow Y$ be a map which is null-homotopic. Prove that f can be extended to a map $g : B^2 \to Y$.

4. Let $f_n : X \to Y$ be a sequence of a continuous functions from the topological space X to a metric space Y. If $\{f_n |\ n \in \mathbb{Z}^+\}$ converges uniformly to f, then f is continuous.

5. Let X be a normal space; A and B be disjoint closed subsets of X. Construct a continuous function $f : X \to [0,1] \subset \mathbb{R}$ such that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$. [You can skip that your f is continuous. Just a construction is required.]

6. Let X be a locally compact Hausdorff space; let the space of continuous functions $\mathcal{C}(X, Y)$ have the compact-open topology. Prove the "evaluation map"

$$e : X \times \mathcal{C}(X, Y) \to Y$$

defined by $e(x, f) = f(x)$ is continuous. Emphasize where the conditions for X are used.

7. Let $\mathbb{H} = \{z = x + iy \in \mathbb{C} \ : \ y > 0\}$ and $\mathrm{SL}(2,\mathbb{R})$ be the group of (2×2)-matrices of determinant $+1$. For $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{R})$ and $z \in \mathbb{H}$, define $A \cdot z = \dfrac{az+b}{cz+d}$.

7A. Prove that this gives an *action* of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{H}$.

7B. Find the *stabilizer* (= isotropy subgroup) of this action at $z = i$.

7C. Is the action *transitive*? Explain why/not.

8. Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering space, and $f : (Y, y_0) \to (X, x_0)$ a continuous map. Then a lift $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ exists if and only if $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

9. Let $p : (\hat{X}, \hat{x}_0) \to (X, x_0)$ be a covering map. Prove that the deck transformation group is isomorphic to $N_{\pi_1(X,x_0)}\Big(p_*(\pi_1(\hat{X}, \hat{x}_0))\Big)$, the normalizer of $p_*(\pi_1(\hat{X}, \hat{x}_0))$ in $\pi_1(X, x_0)$.

10. Let X be the space obtained by the unit disk D in $\mathbb{C} = \mathbb{R}^2$, the boundary identified with itself by the map $f(z) = z^3$; Y a 2-torus. Construct a new space by "connected sum" as follows: Remove a small disk from the interior of D (now part of X), and a small disk from Y, and glue them by a cylinder to make Z. Calculate $\pi_1(Z)$.

11. For a given group $G = \langle g_{\alpha_1}, g_{\alpha_2}, \cdots, g_{\alpha_n} \mid r_{\beta_1}, r_{\beta_2}, \cdots, r_{\beta_m} \rangle$, construct a Cayley complex $\tilde{X}_G$ (a 2-dimensional CW complex) which is simply connected with a properly discontinuous and free (i.e., covering space) action of G. Describe the G-action, and identify the orbit space $G \backslash \tilde{X}_G$.