Qualifying Examination **Real Analysis** **Aug 19, 2009**

Name: ________________________________

Problem 1. [3+3+3+3+3+3 points]
Let μ be a finite measure on $\mathbb{R}$, and the function $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) = \mu((-\infty, x]) \ .$$

In your answers of (a) and (b), please specify explicitly which properties of μ you have used.

(a) Prove that F is non-decreasing (i.e., that $x < y$ implies that $F(x) \leq F(y)$).

(b) Show that F is right-continuous.

(c) If the measure μ is such that $\mu(\{3\}) = 5$ and $\mu(\{8\}) = 4$, and the measure of any set that does not contain 3 and 8 is zero, sketch the function F, indicating all important points on your graph.

(d) The measure μ defined in part (c) defines a distribution. Express this distribution in terms of Dirac delta-distirbutions.

(e) If $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^2$ and μ is the measure defined in part (c), find the value of $\displaystyle\int_{\mathbb{R}} f(x)\,\mathrm{d}\mu(x)$.

Problem 2. [5 points]

Let $f \in L^1_{\text{loc}}(\mathbb{R})$ and let f be continuous at x. Prove that x belongs to the Lebesgue set of f.

Problem 3. [2+3+2+3 points]
Let $X = [0, \infty)$, m be the Lebesgue measure on $[0, \infty)$, and let $\mathrm{d}\mu := f\, \mathrm{d}m$, where

$$f : [0, \infty) \to \mathbb{C} : x \mapsto \mathrm{e}^{-x+ix} \ .$$

(a) Is μ absolutely continuous with respect to m? Why?

(b) Compute $\mu([0, \frac{\pi}{2}])$. You may use the definite intergrals $\int_0^{\pi/2} \mathrm{e}^{-x} \cos x \, \mathrm{d}x = \frac{1}{2}(1 + \mathrm{e}^{-\pi/2})$ and $\int_0^{\pi/2} \mathrm{e}^{-x} \sin x \, \mathrm{d}x = \frac{1}{2}(1 - \mathrm{e}^{-\pi/2})$.

(c) What is $\mathrm{d}|\mu|$? (Just write it down without explanations.)

(d) Compute $|\mu|([0, \frac{\pi}{2}])$.

Problem 4. [5 points]

Consider the function $F : [-1, 1] \to \mathbb{R}$ defined by

$$F(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \in [-1, 0) \cup (0, 1] \ , \\ 0 & \text{if } x = 0 \ . \end{cases}$$

Demonstrate that F is not a function of bounded variation on $[-1, 1]$.

Problem 5. [**4+4+2 points**]

Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions such that $f_n \in L^1(X, \mu)$ for any $n \in \mathbb{N}$, and $f_n \to f$ uniformly on X.

(a) [**4 points**] Assume that $\mu(X) < \infty$, and show that $f \in L^1(X, \mu)$.

(b) **[4 points]** Again, assume that $\mu(X) < \infty$, and show that $\int f_n \, \mathrm{d}\mu \to \int f \, \mathrm{d}\mu$ by using some of the famous convergence theorems.

(c) **[4 points]** Construct an explicit example (using $\mathbb{R}$ and the Lebesgue measure m on it) of a sequence $\{f_n\}_{n=1}^{\infty}$ of $L^1(\mathbb{R}, m)$ functions such that $f_n \to f$ uniformly on $\mathbb{R}$, but $\int f_n \, \mathrm{d}m$ does not converge to $\int f \, \mathrm{d}m$.

Problem 6. [2+2+2+2+2 points]
Consider the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ defined by $f_n = \sqrt{n}\,\chi_{[0,\frac{1}{n}]}$. Let m is the Lebesgue measure on $[0,1]$. Does this function sequence converge (and if yes, then to what limit)

(a) pointwise m-almost everywhere on $\mathbb{R}$?

(b) in $L^1(\mathbb{R})$?

(c) in $L^2(\mathbb{R})$?

(d) in measure?

(e) in $\mathcal{D}'(\mathbb{R})$? (Recall that a sequence of distributions $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{D}'(\mathbb{R})$ converges to $F \in \mathcal{D}'(\mathbb{R})$ if $\langle F_n, \phi\rangle \to \langle F, \phi\rangle$ for any test function $\phi \in \mathcal{D}(\mathbb{R})$.)

Problem 7. [3+2+4 points]

Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of measurable sets in the measure space $(X, \mathcal{M}, \mu)$, and define

$$E := \{x \in X : x \in E_n \text{ for infinitely many } n \in \mathbb{N}\} .$$

(a) **[3 points]** Show that $E = \bigcap_{k \in \mathbb{N}} \bigcup_{n \geq k} E_n$.

(b) **3 points]** Prove that $E \in \mathcal{M}$.

(b) **[4 points]** Use the representation from (a) to prove that if $\sum_{n \in \mathbb{N}} \mu(E_n) < \infty$, then $\mu(E) = 0$.

Problem 8. [5 points]

Demonstrate that the condition that the measure be finite in Egoroff's Theorem is indeed necessary.

Hint: Consider the function sequence $\{f_n\}$ defined by $f_n : [0, \infty) \to \mathbb{R} : x \mapsto \frac{x}{n}$.

Problem 9. [5 points]

Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $\int_{\mathbb{R}} (1+|x|)^{-N} |f(x)| \, \mathrm{d}x \leq C$ for some $N \in \mathbb{N}$ and some positive constant C. Prove that f determines a tempered distribution, i.e., that $|\langle f, \phi \rangle| < \infty$ for any $\phi \in \mathcal{S}$.

Hint: You have to prove that $|\langle f, \phi \rangle| \leq \text{const} \cdot \|\phi\|_{M,k}$ for some M and k, where the seminorms $\| \, \|_{M,k} < \infty$ are defined as $\|\phi\|_{M,k} = \sup_{x \in \mathbb{R}} (1+|x|)^M |\phi^{(k)}(x)|$.

Problem 10. [3+3+4 points]

(a) Give an example of a function in $L^1([0,1], m)$ that does not belong to $L^2([0,1], m)$ (where m is the Lebesgue measure on $[0,1]$).

(b) Give an example of a function in $L^2([0,\infty), m)$ that does not belong to $L^1([0,\infty), m)$ (where m is the Lebesgue measure on $[0,\infty)$).

Hint: The function must be supported on a set of infinite measure – see part (c).

(c) Prove that, if the measure μ of a set E is finite, then if $f \in L^2(E, \mu)$, then $f \in L^1(E, \mu)$.

Problem 11. [3+5 points]

(a) Jensen's inequality holds for probability measures, i.e., positive measures such the measure of the whole space is 1. Generalize Jensen's inequality to any finite measure space $(Y, \mathcal{A}, \nu)$ (i.e., such that $\nu(Y) < \infty$).

(b) Assume that $(X, \mathcal{M}, \mu)$ is a measure space (not necessarily finite), and the function $f \in L^1(X, \mathcal{M}, \mu)$ satisfies $\|f\|_1 = \int_X |f(x)| \, \mathrm{d}\mu(x) = 1$. Prove that

$$\int_E \log |f(x)| \, \mathrm{d}\mu(x) \leq -\mu(E) \log \mu(E)$$

for all subsets $E \in \mathcal{M}$ such that $0 < \mu(E) < \infty$.

Hint: The function $-\log t$ is convex on $(0, \infty)$.

Problem 12. [3+3+6 points]

Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $L^1(\mathbb{R})$ such that $\|f_n - f_{n+1}\|_1 \leq \frac{1}{2^n}$.

(a) Show that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1(\mathbb{R})$.

(b) Does $\{f_n\}_{n\in\mathbb{N}}$ converge in the L^1 norm? Why?

(c) Prove that $f_n \to f$ m-a.e.

Hint: First write $f_n(x) - f(x)$ as a telescoping series and use the triangle inequality to find an upper bound on $|f_n(x) - f(x)|$ in terms of a sum of terms of the form $|f_j(x) - f_{j+1}(x)|$. Then assume that f_n does not converge m-a.e. to f, i.e., there exists a set of a positive measure on which $|f_n(x) - f(x)|$ does not tend to 0 as $n \to \infty$. Use this and the upper bound on $|f_n(x) - f(x)|$ to come to a contradiction.