Qualifying Exam - Algebra, August 2008

Full marks for complete answers to any *six* questions.

Show all work *fully* and *clearly.*

Good luck!

1. Suppose that a group G acts on a set X.

   a. What does it mean to say the action is *transitive*?

   b. Assuming that the action is transitive, suppose that a normal subgroup H of G fixes a point $x_0 \in X$, i.e., $h.x_0 = x_0$, for all $h \in H$. Show that H fixes every point of X, i.e., $h.x = x$, for all $h \in H$ and all $x \in X$.

2. a. Prove that there is no simple group of order 160.

   b. Let G be a finite group with Sylow p-subgroup P (for some prime p). Prove that $N_G(N_G(P)) = N_G(P)$.

3. a. What does it mean to say that a group is i) *solvable*, ii) *nilpotent*?

   b. Let F be a field with more than two elements. Consider the subgroup M of $GL(2, F)$ given by

$$M = \{\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a, b \in F,\ a \neq 0\}.$$

   Show that M is solvable but not nilpotent.

4. a. Let R be a commutative ring and let $\mathfrak{a}$ and $\mathfrak{b}$ be distinct maximal ideals of R. Prove that $R/\mathfrak{a}\mathfrak{b} \cong R/\mathfrak{a} \times R/\mathfrak{b}$ as rings.

   b. Let F be a field with $\operatorname{char} F \neq 2$. By using a, or otherwise, show that $F[x]/(x^2 - 1) \cong F \times F$ as rings.

5. Let R be a ring in which $x^2 = x$, for all $x \in R$.

   a. Prove that R is commutative.

   b. Let $\mathfrak{p}$ be a prime ideal in R. Prove that $R/\mathfrak{p} \cong \mathbb{F}_2$, the finite field with two elements.

6. a. Let F be a finite field and let n be a positive integer. Prove that $F[x]$ contains an irreducible polynomial of degree n.

   b. Determine all primes p for which $x^2 + 1$ is irreducible in $\mathbb{F}_p[x]$ (where $\mathbb{F}_p$ denotes the finite field with p elements).

7. a. Let K/F be a finite Galois extension of fields with Galois group G. Let H be a subgroup of G. Show that there is an $\alpha \in K$ such that $H = \{\sigma \in G : \sigma(\alpha) = \alpha\}$.

   b. Give an example, with justification, of a finite extension of fields that is not separable.

8. a. Show that $\mathbb{Q}(\sqrt{-1}, \sqrt[4]{2})/\mathbb{Q}$ is a Galois extension and determine its Galois group.

   b. Show that $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ is a Galois extension and determine its Galois group.

9. Let K/F be a Galois extension of fields such that $\mathrm{Gal}(K/F) \cong A_4$, the alternating group on 4 letters.

   a. Prove that there is a unique cubic intermediate field, i.e., a unique field K_1 such that $F \subset K_1 \subset K$ and $[K_1 : F] = 3$.

   b. Prove that there is no quadratic intermediate field, i.e., there is no field K_1 such that $F \subset K_1 \subset K$ and $[K_1 : F] = 2$.

10. a. Prove that $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.

    b. Let $\mathbb{Q}^{\times}_{\text{pos}}$ denote the multiplicative group of positive rational numbers. Prove that $\mathbb{Q}^{\times}_{\text{pos}}$ is a free $\mathbb{Z}$-module.