Qualifying Exam - Abstract Algebra, May 2008

Full marks for complete answers to any *six* questions.

Show all work *fully* and *clearly*.

Good luck!

1. a. Let G be a finite p-group (for some prime p). Suppose G acts on a finite set X such that $p \nmid |X|$. Prove that G fixes an element of X, i.e., there is an $x \in X$ such that $g.x = x$, for all $g \in G$.

   b. Let S be a 2-subgroup of the symmetric group S_n with n odd. Show that there is an $i \in \{1, \ldots, n\}$ such that $\sigma(i) = i$, for all $\sigma \in S$.

2. Let G be a nonabelian group of order pq with p, q prime and $p < q$.

   a. Show that p divides $q - 1$.

   b. Show that G has trivial center.

   c. Show that G has $p + ((q-1)/p)$ conjugacy classes.

3. Let $G = SL(2, \mathbb{F}_p)$, the group of 2×2 matrices of determinant one with entries in the finite field $\mathbb{F}_p$ (for p a prime).

   a. Describe a Sylow p-subgroup of G.

   b. Determine the number of Sylow p-subgroups in G.

   c. Determine the number of elements of order p in G.

4. Let R be a ring such that the polynomial ring $R[x]$ is a PID. Prove that R is a field.

5. Let R be a ring in which $x^2 = x$, for all $x \in R$.

   a. Prove that R is commutative.

   b. Let $\mathfrak{p}$ be a prime ideal in R. Prove that $R/\mathfrak{p} \cong \mathbb{F}_2$, the finite field with two elements.

6. Let F denote a splitting field over $\mathbb{Q}$ of $(x^2-2)(x^2-3)$ and let K denote a splitting field over $\mathbb{Q}$ of $(x^2+2)\,(x^2+3)$.

   a. Find $[F : \mathbb{Q}]$ and $[K : \mathbb{Q}]$.

   b. Find explicit generators of F and K over $\mathbb{Q}$, i.e., elements $\alpha \in F$ and $\beta \in K$ such that $F = \mathbb{Q}(\alpha)$ and $K = \mathbb{Q}(\beta)$.

7. Let K/F be a field extension of finite degree.
   a. What does it mean to say that K/F is *normal*? Give an example, with justification, of an extension that is not normal.
   b. What does it mean to say that K/F is *separable*? Give an example, with justification, of an extension that is not separable.

8. a. Prove that $x^p - x - 1 \in \mathbb{F}_p[x]$ is irreducible, for any prime p.
   b. What is the Galois group of $x^p - x - 1$ over $\mathbb{F}_p$?

9. Let K/F be a Galois extension of fields such that $\mathrm{Gal}(K/F) \cong S_4$.
   a. Prove that there is a unique quadratic (i.e., degree two) extension of F contained in K.
   b. Determine the number of cubic (i.e., degree three) extensions of F contained in K. How many of these extensions are normal?

10. a. Prove that $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.
    b. Let $\mathbb{Q}^{\times}_{\mathrm{pos}}$ denote the multiplicative group of positive rational numbers. Prove that $\mathbb{Q}^{\times}_{\mathrm{pos}}$ is a free $\mathbb{Z}$-module.