Topology Qualifying Review Exam
August 13, 2007

• Do all the problems <u>in the right order</u>.

1. Prove/Disprove: Let X be a topological space, $A \subset X$. Suppose $a \in \overline{A}$ (closure of A). Then there exists a sequence of points $\{a_n \in A : n \in \mathbb{Z}^+\}$ which converges to a.

2. Prove/Disprove: A quotient space of a Hausdorff space is Hausdorff.

3. For each $\alpha \in J$, let X_α be a topological space with topology $\mathfrak{T}_\alpha$. Let $X = \prod_{\alpha \in J} X_\alpha$ be given the product topology, and let $\pi_\alpha : X \to X_\alpha$ denote the projection map. Prove that a function $f : Y \to X$ is continuous if each $\pi_\alpha \circ f$ is continuous.

4. Let X be a space and let (Y, d) be a metric space. On the set $\mathcal{C}(X, Y)$ of all continuous functions from X to Y, prove that the compact-open topology is finer than the topology of compact convergence. [The topology of compact convergence has basis

$$\{B_C(f, \epsilon) : \ f \in \mathcal{C}(X, Y),\ C \text{ compact subset of } X,\ \epsilon > 0\},$$

where $B_C(f, \epsilon) = \{g \in \mathcal{C}(X, Y) : \sup_{x \in C} d(f(x), g(x)) < \epsilon\}$].

5. Let X be a paracompact Hausdorff space; let $\mathcal{U} = \{U_\alpha : \alpha \in J\}$ be an indexed open covering of X. Prove that there exists a partition of unity on X dominated by $\mathcal{U}$.

6. Let X be the quotient space of the 2-sphere S^2 obtained by identifying all the points on the equator to a point, and let $Y = S^2 \vee S^2$ be the wedge of two spheres. Prove that X and Y are homeomorphic. What is their fundamental group?

7. A topological property is *hereditary* if every subspace of a space which has the property also has the property. Indicate which of the following properties are hereditary. For non-hereditary properties, give counter-examples: (i) first countability; (ii) second countability; (iii) path connectedness; (iv) metrizability; (v) contractibility.

8. Let X be any topological space; Y a contractible space. Prove that any two continuous maps $f, g : X \to Y$ are homotopic.

9. (a) State Seifert–van Kampen theorem.
(b) Use (a) to calculate the fundamental group of wedge of n circles (use induction).

10. Let $p : (E, e_0) \to (B, b_0)$ be a covering map, where E and B are path-connected. Let $H_0 = p_*(\pi_1(E, e_0))$. Define a natural map $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ and prove that ϕ induces a bijective map $\Phi : \pi_1(B, b_0)/H_0 \to p^{-1}(b_0)$.

11. Calculate the fundamental group of $(T \# P) \vee S$, where T is the torus; P is the projective plane; and S is the 2-sphere; $\#$ denotes the connected sum, and $\vee$ denotes the wedge.

12. The group $E(2) = \mathbb{R}^2 \rtimes \mathrm{O}(2)$ (where $\mathrm{O}(2)$ is the orthogonal group) is $\mathbb{R}^2 \times \mathrm{O}(2)$ as sets, but the group operation is given by $(a, A) \cdot (b, B) = (a + Ab, AB)$. Then $E(2)$ acts on the space $\mathbb{R}^2$ by $(a, A) \cdot x = a + Ax$ for $x \in \mathbb{R}^2$. Let π be the subgroup of $E(2)$ generated by the 3 elements $(e_1, I), (e_2, I)$ and $(\mathbf{0}, A)$, where I is 2×2 identity matrix, and

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ and } A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(a) Show π contains a subgroup $\mathbb{Z}^2$ of index 2.
(b) Is the action of π on $\mathbb{R}^2$ free?
(c) Identify the orbit space $\mathbb{R}^2/\pi$. Explain how you derive the conclusion.

End