Topology Qualifying Exam
August 2006

Give clear and complete arguments for each of the problems, but try to avoid giving excessive detail. You may use major theorems when appropriate. When doing so, either name or state carefully the theorem. Try to concentrate your efforts on whole problems. One complete answer is better than two half-complete answers. In all cases, justify your answers.

Part 1. Do six of these problems. Please make clear which six you wish to be graded.

1. Show that a space X is Hausdorff if and only if the *diagonal* $\Delta = \{(x, x) \mid x \in X\}$ is closed in $X \times X$.

2. Given a product space $X = \Pi_{\alpha \in J} X_\alpha$ with the product topology, let $\pi_\alpha \colon X \to X_\alpha$ be projection onto the α-coordinate. If $f \colon A \to X$ is a function, show that f is continuous if and only if $\pi_\alpha \circ f \colon A \to X_\alpha$ is continuous for every $\alpha \in J$.

3. (a) What is the universal cover of the torus $T^2 = S^1 \times S^1$? (no proof required)
(b) Prove that every continuous map $S^2 \to T^2$ is homotopic to a constant map.
(c) Find a map $S^1 \to T^2$ that is not homotopic to a constant map.

4. (a) Show that if Y is Hausdorff then the space of continuous maps $\mathscr{C}(X, Y)$ with the compact-open topology is Hausdorff.
(b) Consider the sequence of functions $f_n \in \mathscr{C}(\mathbb{R}, \mathbb{R})$ given by $f_n(x) = nx$. Does this sequence converge in the compact-open topology? Explain why or why not.

5. Let $p \colon \widetilde{X} \to X$ be a covering map with $\widetilde{X}$ path connected.
(a) Given $x_0 \in X$, define the *lifting correspondence* $\phi \colon \pi_1(X, x_0) \to p^{-1}(x_0)$.
(b) Show that ϕ is surjective.
(c) Show that if $\widetilde{X}$ is simply connected then ϕ is injective.

6. Recall that a subspace $A \subset X$ is a *retract* if there is a retraction $r \colon X \to A$ (a continuous map such that $r \circ i = \mathrm{id}_A$, where $i \colon A \to X$ is the inclusion map). Prove that a retract of a contractible space is contractible.

7. Let X be a space with a countable basis.
(a) Prove that every subset $A \subset X$ has a countable dense subset (recall that a subset of Y is *dense* if its closure is equal to Y).
(b) Prove that if $A \subset X$ is uncountable then uncountably many points of A are limit points of A.

Part 2. Do four of these problems. Please make clear which four you wish to be graded.

8. Construct a covering space of the figure eight (two circles joined at one point) and use it to prove that the fundamental group of the figure eight is not abelian.

9. Let X be a non-compact, locally compact Hausdorff space. Prove that X has a one-point compactification.

10. Let $X \subset \mathbb{R}^2$ be the union of the unit circle and its vertical and horizontal diameters. Compute the fundamental group of X.

11. Let G be the free group of rank 2. Find two subgroups H_1 and H_2 of index 4 in G that are not conjugate. Give generating sets for these groups and explain why they have the desired properties.

12. Prove (from basic definitions) that if X and Y are compact then so is $X \times Y$.