Topology Qualifying Exam

May 2006

Give clear and complete arguments for each of the problems, but try to avoid giving excessive detail. You may use major theorems when appropriate. When doing so, either name or state carefully the theorem. Try to concentrate your efforts on whole problems. One complete answer is better than two half-complete answers. In all cases, justify your answers.

Part 1. Do six of these problems. Please make clear which six you wish to be graded.

1. Define a function $f\colon \mathbb{R} \to \mathbb{R}^\omega$ by the equation $f(t) = (t, \frac{1}{2}t, \frac{1}{3}t, \frac{1}{4}t, \frac{1}{5}t, \ldots)$. If $\mathbb{R}^\omega$ is given the box topology, is f continuous?

2. Let $X = \bigcup_{i=1}^\infty X_i$ where each X_i is a simply connected subspace of X and $X_i \subset X_{i+1}$ for all i.
(a) Show that if each X_i is open, then X is simply connected.
(b) Show that if each X_i is a closed set, then X may fail to be simply connected.

3. Let X be a locally compact Hausdorff space and let $\mathscr{C}(X, Y)$ be the set of continuous maps from X to Y, with the compact-open topology. Define the *evaluation map* $e\colon X \times \mathscr{C}(X, Y) \to Y$ by $e(x, f) = f(x)$. Prove that the evaluation map is continuous.

4. Let $X = I \times I$ with the dictionary order topology (the *ordered square*). Show that X is not locally path connected. What are the path components of this space?

5. For each pair of spaces (X, A) below, determine whether there exists a retraction $r\colon X \to A$ (a continuous map such that $r \circ i = \mathrm{id}_A$ where $i\colon A \hookrightarrow X$ is inclusion).
(a) $X = \mathbb{R}$ and $A = [0, 1]$.
(b) X is the closed unit disk in $\mathbb{R}^2$ and A is its boundary circle.
(c) X is the Möbius band and A is its boundary circle.

6. In the plane $\mathbb{R}^2$ identify the x-axis to a point and give the resulting set X the quotient topology.
(a) Is X Hausdorff?
(b) Is X regular?
(c) Does the projection $(x, y) \mapsto y$ induce a continuous map $X \to \mathbb{R}$?

7. Let $p\colon \widetilde{X} \to X$ be a covering space such that $p^{-1}(x)$ consists of two points for every $x \in X$. Show that there is a continuous map $f\colon \widetilde{X} \to \widetilde{X}$ such that $p \circ f = p$ (i.e. a *covering translation*), which is not the identity map.

Part 2. Do four of these problems. Please make clear which four you wish to be graded.

8. (a) Prove that a finite product of locally compact spaces is locally compact.
(b) Prove that if the product space $\prod_{\alpha \in A} X_\alpha$ is locally compact, then every factor X_α is locally compact, and all but finitely many of the factors are compact.

9. Prove that if X and Y are connected then $X \times Y$ is connected.

10. (a) Give an outline of the proof that every subgroup of a free group is free.
(b) If X is a finite connected graph, find the rank of its fundamental group in terms of the numbers of vertices and edges.
(c) If G is free of rank 2 and $H \subset G$ is a subgroup of index k, determine the rank of H.

11. Let G be the free group of rank 2. Find all subgroups of G of index 2, up to conjugacy. Give generating sets for these subgroups.

12. Let $f\colon X \to Y$ be a continuous map. The *mapping cylinder* M_f is the quotient space of the disjoint union $X \times [0,1] \sqcup Y$ by the equivalence relation $(x,1) \sim f(x)$ for every $x \in X$. (That is, "glue" $X \times \{1\}$ to Y using f.) Prove that the "inclusion map" $i\colon Y \hookrightarrow X \times [0,1] \sqcup Y \to M_f$ is a homotopy equivalence. That is, find a homotopy inverse $g\colon M_f \to Y$ such that $i \circ g \simeq \mathrm{id}_{M_f}$ and $g \circ i \simeq \mathrm{id}_Y$.