Department of Mathematics
M.A. Comprehensive/Ph.D. Qualifying Exam in Analysis
Spring 2006

1. *(a)* Show that if $\int_E f^2 \, dx \leq 1$, then $\int_E |f| \, dx \leq |E|^{1/2}$.
(b) Let f_j be functions defined on a set E of finite measure. Assume that f_j converges pointwise almost everywhere to a function f on E, and that $\int_E f_j^2 \, dx \leq 1$ for all j. Show that

$$\lim_{j \to \infty} \int_E |f_j - f| \, dx = 0.$$

2. Let f be the Cantor-Lebesgue function on $[0,1]$. (Thus f is strictly increasing on the Cantor set C and f is constant on each of the intervals in $[0,1] - C$.) Let the function $g : [0,1] \longrightarrow [0,1]$ be defined by setting $g(y)$ equal to the smallest number x such that $f(x) = y$.

(a) Show that g is measurable.
(b) Show that there exists a measurable set $E \subset [0,1]$ such that $g^{-1}(E)$ is not measurable. (You may assume the existence of a non-measurable set in $[0,1]$.)

3. Show that

$$\lim_{n \to \infty} \int_0^\infty \frac{e^{-x/n} \sin(x/n)}{1 + x^2} \, dx = 0.$$

4. Suppose $K(x,y)$ is a measurable function on $\mathbf{R}^2$, and $f(y)$ is a measurable function on $\mathbf{R}$.
(a) Show that for all $x \in \mathbf{R}$, if $1 < p < \infty$ then

$$\int_{-\infty}^\infty |K(x,y)f(y)| \, dy \leq \left[\int_{-\infty}^\infty |K(x,y)||f(y)|^p \, dy \right]^{1/p} \left[\int_{-\infty}^\infty |K(x,y)| \, dy \right]^{1/p'} .$$

(b) Suppose there exists a constant $C > 0$ such that

$$\int_{-\infty}^\infty |K(x,y)| \, dx \leq C \quad \text{for all } x \in \mathbf{R}$$

and

$$\int_{-\infty}^\infty |K(x,y)| \, dy \leq C \quad \text{for all } y \in \mathbf{R}.$$

Show that if $f \in L^p$ $(1 < p < \infty)$, and $g(x)$ is defined for $x \in \mathbf{R}$ by

$$g(x) = \int_{-\infty}^\infty K(x,y) f(y) \, dy,$$

then the integral defining g exists for a.e. x, and $g \in L^p$.

5. Suppose f and g are functions on $\mathbf{R}^n$ such that for all $\alpha \in \mathbf{R}$,

$$|\{x : |f(x)| > \alpha\}| = |\{x : |g(x)| > \alpha\}|.$$

Show that $\|f\|_p = \|g\|_p$ for all $0 < p < \infty$.

6. Let $\{\phi_k\}$ be an orthonormal basis for L^2, and let $\{b_k\}$ be a bounded sequence of numbers. Show that for every $f \in L^2$ there exists $g \in L^2$ such that $\langle g, \phi_k \rangle = b_k \langle f, \phi_k \rangle$ for all k.

7. Suppose $\{\phi_i(x)\}_{i \in \mathbf{N}}$ is an orthonormal basis for $L^2(\mathbf{R})$. For each ordered pair $(i, j) \in \mathbf{N} \times \mathbf{N}$, define w_{ij} on $\mathbf{R}^2$ by $w_{ij}(x, y) = \phi_i(x)\phi_j(y)$. Prove that $\{w_{ij}\}_{(i,j) \in \mathbf{N} \times \mathbf{N}}$ is an orthonormal basis for $L^2(\mathbf{R} \times \mathbf{R})$.

8. Let $\{f_n\}$ be a sequence of absolutely continuous functions on $[a, b]$. Suppose that f_n converges pointwise to a function f on $[a, b]$, and f_n' converges in L^1 norm to a function g on $[a, b]$. Prove that f is absolutely continuous on $[a, b]$ and that $f' = g$ a.e. on $[a, b]$.

9. Let Σ be a σ-algebra of subsets of S, and let μ be a measure on Σ. Define Σ_0 to be the collection of all subsets B of S such that for some set A in Σ and some sets Z_1 and Z_2 in Σ with $\mu(Z_1) = \mu(Z_2) = 0$,

$$A - Z_1 \subset B \subset A \cup Z_2.$$

Show that Σ_0 is a σ-algebra of subsets of S.