Department of Mathematics
PhD Qualifying Exam in Analysis (MATH 5453-5463)
May 2010

Directions: Work as many problems as you can; there are 100 total points. If you are asked to “state” a theorem, then no proof is expected unless it is asked for explicitly. You have three hours to complete this exam.

Notation: Unless specified otherwise, $L^p(\mu)$ (for $1 \leq p \leq \infty$) denotes the space of $L^p$ functions on an abstract measure space $(X, \mathcal{M}, \mu)$. We use notations such as $L^p([a, b])$, $L^p([a, \infty)$, and $L^p(\mathbb{R})$ to denote the space of $L^p$ functions on the indicated interval with respect to the standard Lebesgue measure, which we denote by $\mu$. Integrals with respect to the standard Lebesgue measure may be denoted by either $\int_{[a,b]} f \, d\mu$ or $\int_{a}^{b} f(t) \, dm(t)$. We also use $C([a,b])$ to denote the set of continuous real-valued functions on the interval $[a,b]$.

(1) Consider the metric spaces $(C([0,1]), d_1)$ and $(C([0,1]), d_2)$, where the metrics $d_1$ and $d_2$ are defined by

$$d_1(f, g) = \int_{0}^{1} |f(x) - g(x)| \, dx, \quad d_2(f, g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}, \quad (\text{for } f, g \in C([0,1])).$$

Also let $I : C([0,1]) \to C([0,1])$ denote the identity mapping $I(f) = f$.

(a) [5pts] Is the mapping $I : (C([0,1]), d_1) \to (C([0,1]), d_2)$ continuous? Justify your answer by either a proof (if the assertion is true) or a counterexample (if the assertion is false).

(b) [5pts] Is the mapping $I : (C([0,1]), d_2) \to (C([0,1]), d_1)$ continuous? Justify your answer by either a proof (if the assertion is true) or a counterexample (if the assertion is false).

(c) [4pts] One of the metric spaces $(C([0,1]), d_1)$ and $(C([0,1]), d_2)$ is complete and the other is not. Which is which? Explain your answer, but you don’t have to provide any detailed proofs.

(2) Let $(r_n)$ be an enumeration of the rational numbers $\mathbb{Q}$ and let $A \subseteq \mathbb{R}$ be the set defined by

$$A = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \left( r_n - \frac{1}{m2^{n+1}}, r_n + \frac{1}{m2^{n+1}} \right).$$

(a) [3pts] Explain why $A$ is a Lebesgue measurable set and compute its Lebesgue measure $m(A)$.

(b) [3pts] With respect to the complete metric space $\mathbb{R}$ (usual metric understood) is $A$ a first category set or a second category set? Explain.

(c) [3pts] Is $A = \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \left( r_n - \frac{1}{m2^{n+1}}, r_n + \frac{1}{m2^{n+1}} \right)$? Explain.

(3) (a) [3pts] State Fatou’s lemma for the Lebesgue integral.

(b) [8pts] Let $(X, \mathcal{M}, \mu)$ be a measure space, let $1 \leq p \leq \infty$, and let $\{f_n : X \to \mathbb{R} \mid n \in \mathbb{N}\}$ be a sequence of functions in $L^p(\mu)$ such that for some real number $M > 0$ we have $\|f_n\|_p \leq M$ for every $n \in \mathbb{N}$. If $(f_n)$ converges in measure to a measurable function $f : X \to \mathbb{R}$, then prove that $f \in L^p(\mu)$.

(c) [3pts] For the situation in part (b) can we also conclude that $f_n \to f$ in the $L^p$ norm? Explain.

(4) [8pts] If $(X, \mathcal{M}, \mu)$ is a measure space and if $f \in L^1(\mu)$, then prove that for every $\epsilon > 0$ there exists a measurable set $F \in \mathcal{M}$ such that $\mu(F) < \infty$ and $\int_{X \setminus F} |f| \, d\mu < \epsilon$.

(5) [8pts] Prove that if $f \in L^1([a,b])$ has the property that $\int_{a}^{b} f(t) \, dm(t) = 0$ for every $x \in [a,b]$, then $f = 0$ a. e. on $[a,b]$.

(continued on next page)
(6) (a) [3pts] State Hölder’s inequality.
(b) [5pts] Let \((X, \mathcal{M}, \mu)\) be a measure space and let \(r, s \in \mathbb{R}\) be such that \(r > 1\) and \(s > 1\) (note that \(r\) and \(s\) are otherwise arbitrary). Find a positive real number \(\alpha\) for which the following assertion is true.
\[
f \in L^r(\mu) \text{ and } g \in L^s(\mu) \Rightarrow |fg|^{\alpha} \in L^1(\mu)
\]
(your answer will depend on \(r\) and \(s\)).

(7) [8pts] Let \(f : [a, b] \rightarrow \mathbb{R}\) be a function that is continuous at every point of the closed interval \([a, b]\), differentiable at every point of the open interval \((a, b)\), and for which there exists a real number \(M > 0\) such that \(|f'(x)| \leq M\) for every \(x \in (a, b)\). Prove that \(f\) is absolutely continuous on \([a, b]\).

(8) Let \((X, \mathcal{M}, \mu)\) and \((Y, \mathcal{N}, \nu)\) be \(\sigma\)-finite measure spaces, and let \((X \times Y, \mathcal{M} \times \mathcal{N}, \mu \times \nu)\) be the product measure space.
(a) [3pts] If \(E \in \mathcal{M} \times \mathcal{N}\), then express the product measure \((\mu \times \nu)(E)\) in terms of two equivalent integrals, where one integral is with respect to the measure \(\mu\) and the other integral is with respect to the measure \(\nu\) (this is part of the statement of the product measure theorem, but no proofs are being asked for here).
(b) [7pts] Prove that if \(\lambda : \mathcal{M} \rightarrow [0, \infty]\) and \(\eta : \mathcal{N} \rightarrow [0, \infty]\) are \(\sigma\)-finite measures for which \(\lambda \ll \mu\) and \(\eta \ll \nu\), then \(\lambda \times \eta \ll \mu \times \nu\).

(9) This problem deals with the “counting measure space” \((\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)\), where \(\mathcal{P}(\mathbb{N})\) is the power set of \(\mathbb{N}\) and \(\mu\) is the counting measure.
(a) [4pts] Describe the space \(L^p(\mu)\) for \(1 \leq p < \infty\) (be as specific as possible).
(b) [3pts] Give an example of a function \(f : \mathbb{N} \rightarrow \mathbb{R}\) such that \(f \in L^2(\mu)\), but \(f \notin L^1(\mu)\).
(c) [6pts] Prove that \(L^1(\mu) \subseteq L^2(\mu)\).

(10) (a) [3pts] State the Riesz Representation Theorem for \(L^p(\mu)\).
(b) [5pts] For the counting measure space \((\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)\) (see Problem (9)) show that \(\Lambda : L^2(\mu) \rightarrow \mathbb{R}\) defined by
\[
\Lambda(f) = \sum_{n=1}^{\infty} \frac{f(n)}{n} \quad (f \in L^2(\mu))
\]
is a bounded linear functional on \(L^2(\mu)\) and compute the norm of \(\Lambda\).