MA\QUALIFIER EXAM, TOPOLOGY, SP 2008 (52 PTS)

NAME ________________________________

I. BASICS.
1. Let $X$ be a space, $A \subset B \subset X$, and $x \in A'$. Show that $x \in B'$. (3 pts)

2. Suppose that $f : X \rightarrow Y$ is a function between spaces $X$ and $Y$ and $f$ is continuous at $x$ for all $x \in X$. Prove that for each open subset $V$ of $Y$, $f^{-1}(V)$ is open in $X$. (3 pts)

3. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be maps. Prove that $g \circ f : X \rightarrow Z$ is a map. (3 pts)

4. Let $A$ be a closed subset of a space $X$, suppose that $(x_n)$ is a sequence in $A$, and $(x_n)$ converges to an element $x$ of $X$. Show that $x \in A$. (3 pts)

5. Let $X$ be a compact space and $f : X \rightarrow Y$ a surjective map. Prove that $Y$ is compact. (3 pts)

6. Suppose that $A$ is a connected subspace of a space $X$. Show that $\overline{A}$ is connected. (3 pts)

7. Let $X$, $Y$ be nonempty spaces where $Y$ is compact. Prove that the coordinate projection $\pi_X : X \times Y \rightarrow X$ is a closed map. (You may use the Tube Lemma). (3 pts)

8. Suppose that $X$ is a space, $f : X \rightarrow Y$ is a surjective function, and $Y$ is given the quotient topology induced by the function $f$. Prove that $f$ is a map. Let $Z$ be a space and $g : Y \rightarrow Z$ a function. Prove that $g$ is a map iff $g \circ f$ is a map. (3 pts)

II. MORE ADVANCED.
1. Let $f : X \rightarrow Y$ be a map between spaces $X$ and $Y$ and suppose that $A$ is a $G_\delta$-subset of $Y$. Show that $f^{-1}(A)$ is a $G_\delta$-subset of $X$. (3 pts)

2. Let $X$ be a space and $\mathcal{F}$ a locally finite collection of closed subsets of $X$. Prove that $\bigcup \mathcal{F}$ is closed in $X$. (3 pts)

3. Let $X$ be a regular space and $\mathcal{B}$ a base for its topology. Suppose that for each pair $U$, $V \in \mathcal{B}$ with $\overline{U} \subset V$, there is a map $f_{U,V} : X \rightarrow [0,1]$ with $f_{U,V}(\overline{U}) \subset \{0\}$ and $f_{U,V}(X \setminus V) \subset \{1\}$. Prove that $\{f_{U,V} | U, V \in \mathcal{B}, \overline{U} \subset V \}$ separates points and closed sets. (We will use this definition. A collection $\mathcal{F}$ of maps of $X$ to $[0,1]$ separates points and closed sets if for each $x \in X$ and closed subset $A$ of $X$ with $x \notin A$, there exists $f \in \mathcal{F}$ with $f(x) = 0$ and $f(A) \subset \{1\}$.) (3 pts)

4. Let $X$ be a locally compact space, $Y$ a space, and $f : X \rightarrow Y$ a function. Prove that $f$ is a map if $f|K : K \rightarrow Y$ is a map for each compact subset $K$ of $X$. (3 pts)

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5. Let $X$ be a space and $\{Y_\alpha | \alpha \in \Gamma\}$ an indexed collection of spaces. Suppose that $f : X \to \prod \{Y_\alpha | \alpha \in \Gamma\}$ is a function. Prove that $f$ is a map iff $\pi_\alpha \circ f : X \to Y_\alpha$ is a map for all $\alpha \in \Gamma$. (3 pts)

6. Let $(X, d)$ be a complete metric space. Suppose that $(D_n)$ is a sequence of closed subsets of $X$ such that for each $n \in \mathbb{N}$,

   (1) $D_{n+1} \subset D_n$, and
   (2) there exists $x_n \in D_n$ such that $D_n \subset B_d(x_n, \frac{1}{2^n})$.

Prove that $(x_n)$ is a Cauchy sequence, $(x_n)$ has a limit $x \in X$, and $x$ is an element of $D_n$ for all $n \in \mathbb{N}$. (3 pts)

7. Suppose that $B$ is a space, $S$ is a subspace of $B$, and $a \in S$. Assume that $\pi_1(B, a)$ is a finite group and $\pi_1(S, a)$ is an infinite group. Prove that there does not exist a retraction $r$ of $B$ onto $S$. (3 pts)

8. Let $X$ be a set and $d$ the discrete metric on $X$. Prove that $(X, d)$ is a complete metric space. (3 pts)

9. This problem deals with Baire spaces and complete metrics.

   (a) Give an example of a countably infinite subset $S$ of $\mathbb{R}$ so that with the standard metric $d$, $(S, d)$ is a complete metric space. Prove your claim. Is your example a Baire space? Explain. (2 pts)

   (b) Show that $\mathbb{Q}$ with its standard topology is not a Baire space. Explain why $\mathbb{Q}$ cannot be given a complete metric that induces its topology. (2 pts)

End of Exam. Please turn in this sheet with your solutions all stapled together.