1. Suppose that a group $G$ acts on a set $X$.
   a. What does it mean to say the action is transitive?
   
   b. Assuming that the action is transitive, suppose that a normal subgroup $H$ of $G$ fixes a point $x_0 \in X$, i.e., $h.x_0 = x_0$, for all $h \in H$. Show that $H$ fixes every point of $X$, i.e., $h.x = x$, for all $h \in H$ and all $x \in X$.

2. a. Prove that there is no simple group of order 160.
   
   b. Let $G$ be a finite group with Sylow $p$-subgroup $P$ (for some prime $p$). Prove that $N_G(N_G(P)) = N_G(P)$.

3. a. What does it mean to say that a group is i) solvable, ii) nilpotent?
   
   b. Let $F$ be a field with more than two elements. Consider the subgroup $M$ of $GL(2, F)$ given by
   
   $$ M = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a, b \in F, a \neq 0 \right\}. $$
   
   Show that $M$ is solvable but not nilpotent.

4. a. Let $R$ be a commutative ring and let $a$ and $b$ be distinct maximal ideals of $R$. Prove that $R/a b \cong R/a \times R/b$ as rings.
   
   b. Let $F$ be a field with char $F \neq 2$. By using a, or otherwise, show that $F[x]/(x^2 - 1) \cong F \times F$ as rings.

5. Let $R$ be a ring in which $x^2 = x$, for all $x \in R$.
   
   a. Prove that $R$ is commutative.
   
   b. Let $p$ be a prime ideal in $R$. Prove that $R/p \cong \mathbb{F}_2$, the finite field with two elements.

6. a. Let $F$ be a finite field and let $n$ be a positive integer. Prove that $F[x]$ contains an irreducible polynomial of degree $n$.
   
   b. Determine all primes $p$ for which $x^2 + 1$ is irreducible in $\mathbb{F}_p[x]$ (where $\mathbb{F}_p$ denotes the finite field with $p$ elements).
7. a. Let $K/F$ be a finite Galois extension of fields with Galois group $G$. Let $H$ be a subgroup of $G$. Show that there is an $\alpha \in K$ such that $H = \{ \sigma \in G : \sigma(\alpha) = \alpha \}$.

b. Give an example, with justification, of a finite extension of fields that is not separable.

8. a. Show that $\mathbb{Q}(\sqrt{-1}, \sqrt[4]{2})/\mathbb{Q}$ is a Galois extension and determine its Galois group.

b. Show that $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ is a Galois extension and determine its Galois group.

9. Let $K/F$ be a Galois extension of fields such that $\text{Gal}(K/F) \cong A_4$, the alternating group on 4 letters.

a. Prove that there is a unique cubic intermediate field, i.e., a unique field $K_1$ such that $F \subset K_1 \subset K$ and $[K_1 : F] = 3$.

b. Prove that there is no quadratic intermediate field, i.e., there is no field $K_1$ such that $F \subset K_1 \subset K$ and $[K_1 : F] = 2$.

10. a. Prove that $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.

b. Let $\mathbb{Q}_{\text{pos}}^\times$ denote the multiplicative group of positive rational numbers. Prove that $\mathbb{Q}_{\text{pos}}^\times$ is a free $\mathbb{Z}$-module.