Do all the problems in the right order.

1. Let \( Y \) be a Hausdorff space; and \( f, g : X \to Y \) be continuous maps. Suppose \( f = g \) on a subset \( A \) of \( X \) which is dense in \( X \). Prove that \( f = g \) on \( X \).

2. Let \( \{x_n : n = 1, 2, 3, \cdots \} \) be a sequence of points in a topological space \( X \), converging to \( x_0 \). Prove that the set \( K = \{x_n : n = 0, 1, 2, 3, \cdots \} \) is compact.

3. For each \( \alpha \in J \), let \( X_\alpha \) be a topological space with topology \( \mathcal{T}_\alpha \). Let \( X = \prod_{\alpha \in J} X_\alpha \) be given the product topology, and let \( \pi_\alpha : X \to X_\alpha \) denote the projection map. Prove that a function \( f : Y \to X \) is continuous if each \( \pi_\alpha \circ f \) is continuous.

4. Let \( C \) be the set of all continuous real-valued functions on \([0, 1]\). For each \( f \in C \) and \( \epsilon > 0 \), define

\[
M(f, \epsilon) = \{g \in C : \int_0^1 |f - g| < \epsilon\}
\]

\[
U(f, \epsilon) = \{g \in C : \sup_{x \in [0,1]} |f(x) - g(x)| < \epsilon\}.
\]

1. Prove that \( \mathcal{M} = \{M(f, \epsilon) : f \in C, \epsilon > 0\} \) forms a basis for a topology.
2. Compare the two topologies generated by \( \mathcal{M} \) and \( \mathcal{U} = \{U(f, \epsilon) : f \in C, \epsilon > 0\} \).

5. Let \( \mathcal{A} = \{A_\alpha : \alpha \in J\} \) be a locally-finite collection of closed covering of a space \( X \). Let \( f : X \to Y \) be a function, and suppose \( f|_{A_\alpha} \) (restriction of \( f \) to \( A_\alpha \)) is continuous for each \( \alpha \in J \). Prove that \( f \) is continuous.

6. Let \( D \) be the unit disk in \( \mathbb{R}^2 \) with the subspace topology. We identify all the boundary points of \( D \) (to one point \( p \)), and give the quotient topology to the resulting quotient set \( Q \). Prove that \( Q \) is homeomorphic to the sphere \( S^2 \). Pay special attention to the local neighborhood system of \( Q \) at the point \( p \).

7. Let \( X \) be a locally compact Hausdorff space. For a space \( Y \), the set of all continuous maps from \( X \) to \( Y \) is denoted by \( C(X, Y) \). It has the compact-open topology. Prove the map \( e : X \times C(X, Y) \to Y \) defined by

\[
e(x, f) = f(x)
\]

is continuous.

8. Given a path \( f \) in a space \( X \) from \( x_0 \) to \( x_1 \), let \( \overline{f} \) be the path in \( X \) defined by \( \overline{f}(s) = f(1 - s) \). Prove that \( f + \overline{f} \) is homotopic to the constant path \( e_{x_0} \) at \( x_0 \).

9. (a) State Seifert–van Kampen theorem (classical version, to be applied to the next question).
   (b) Use (a) to calculate the fundamental group of the following space: A space \( A \) is a torus with an open disk \( D \) removed. Let \( f : \partial B \to \partial A \) be a map from the boundary of a 2-ball \( B \) to the boundary of \( A \) winding twice (that is, double covering map from a circle to a circle). Let \( X \) be the space joining the 2-cell \( B \) by the map \( f \). What is the fundamental group of \( X \)?

10. Let \( p : (E, e_0) \to (B, b_0) \) be a covering map. Let \( f : (Y, y_0) \to (B, b_0) \) be a continuous map. Suppose \( Y \) is path connected and locally path connected. If \( f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(E, e_0)) \), the \( f \) can be lifted to a map \( \tilde{f} : (Y, y_0) \to (E, e_0) \).
11. Let \( p : X \to B \) be a regular covering map; let \( G \) be its group of covering transformations (so that the action of \( G \) is properly discontinuous). Let \( \pi : X \to X/G \) be the projection map. Show that there is a homeomorphism \( k : X/G \to B \) such that \( k \circ \pi = p \).

\[ \begin{array}{ccc}
X & \xrightarrow{\pi} & X \\
\downarrow & & \downarrow \pi \\
X/G & \xrightarrow{k} & B 
\end{array} \]

12. The group \( E(2) = \mathbb{R}^2 \rtimes O(2) \) (where \( O(2) \) is the orthogonal group) is \( \mathbb{R}^2 \times O(2) \) as sets, but the group operation is given by \( (a, A) \cdot (b, B) = (a + Ab, AB) \).

(a) \( E(2) \) acts on the space \( \mathbb{R}^2 \) by \( (a, A) \cdot x = a + Ax \) for \( x \in \mathbb{R}^2 \). Show this actually defines an action.

(b) Let \( \pi \) be the subgroup of \( E(2) \) generated by the 3 elements \((e_1, I), (e_2, I)\) and \((a, A)\), where

\[ e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad a = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \]

Show \( \pi \) contains a subgroup \( \mathbb{Z}^2 \) of index 2.

(c) Is the action of \( \pi \) on \( \mathbb{R}^2 \) free?

(d) Identify the orbit space \( \mathbb{R}^2/\pi \). Explain how you derive the conclusion.