CALCULUS I FALL 1998

TEST 3

In order to get full credit, all answers must be accompanied by appropriate justifications.

Outside of a dog, a book is man's best friend; inside a dog, it is too dark to read.

GROUCHO MARX

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1. Use linear and quadratic approximation to estimate the value of $\sqrt{63}$.

(20 points)

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Put $f(x) = \sqrt{x}$, then $f'(x) = 1/(2\sqrt{x})$ and $f''(x) = -1/(4\sqrt{x^3})$. Then $f(64) = 8$, $f'(64) = 1/16$, $f''(64) = -1/2048$. So the linear approximation is

$$\ell(x) = f(64) - f'(64) = 8 - \frac{1}{16} = 7.9375$$

and the quadratic approximation is

$$q(x) = f(64) - f'(64) + \frac{1}{2}f''(64) = 8 - \frac{1}{16} - \frac{1}{4096} \simeq 7.9372\ldots$$

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2. Use Newton's method to find the solution of $\tan x = x$ in the interval $(\pi/2, 3\pi/2)$; give an answer which is accurate to three decimal places.

(20 points)

. . . .

Put $f(x) = \tan x - x$ then $f'(x) = \sec^2 x - 1 = \tan^2 x$ and so Newton's process requires us to use the algorithm

$$x_{n+1} = x_n - \frac{\tan x_n - x_n}{\tan^2 x_n}.$$

Starting with $x_1 = 2$ we get $x_2 = 6.120$, $x_3 = 238$ which appears to be diverging. A look at the graph shows a root close to 4 but very steep tangents. Starting with $x_1 = 4.5$, we get $x_2 = 4.494$, $x_3 = 4.493 = x_4$. Consequently our estimate is 4.493 to three decimal places.

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3. Show that the equation $6x^5 + 13x + 1 = 0$ has exactly one real root.

(20 points)

. . . .

Put $f(x) = 6x^5 + 13x + 1$, then f is continuous and differentiable everywhere, so it satisfies all the hypotheses of the mean value theorem and the intermediate value theorem. Also $f'(x) = 30x^4 + 13 \geqslant 13$, for all x. If f had two or more roots then the mean value theorem would imply that f' has at least one root - but it has no real roots. Hence f has at most one root. Also $f(-1) = -18$, $f(0) = 1$. Consequently, the intermediate value theorem shows that there is at least one root between -1 and 0. Hence f has exactly one root.

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4. Find the absolute maximum and minimum of $f(x) = \sin 2x - x$ on the interval $[0, \pi]$.

(20 points)

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Put $f(x) = \sin 2x - x$ and so $f'(x) = 2\cos 2x - 1$. Consequently, the critical points occur when $\cos 2x = 1/2$. Notice that, since $0 \leqslant x \leqslant \pi$ we have $0 \leqslant 2x \leqslant 2\pi$ and so the critical points occur when $2x = \pi/3,\ 5\pi/3$; that is, $x = \pi/6,\ 5\pi/6$. Now $f(0) = 0$, $f(\pi/6) = \sqrt{3}/2 - \pi/6$, $f(5\pi/6) = -\sqrt{3}/2 - 5\pi/6$, $f(\pi) = -\pi$. Consequently, the absolute maximum is $f(\pi/6) = \sqrt{3}/2 - \pi/6$ and the absolute minimum is $f(5\pi/6) = -\sqrt{3}/2 - 5\pi/6$.

5. Sketch the graph of $f(x) = \cos^3 x + 6\cos x$ on the interval $[0, 2\pi]$. Indicate the intervals on which it is monotonic and those on which it is concave up or down.

(20 points)

. . . .

Put $f(x) = \cos^3 x + 6\cos x$, then $f'(x) = -3\cos^2 x \sin x - 6\sin x = -3\sin x(\cos^2 x + 2)$ and

$$f''(x) = -3\cos x(\cos^2 x + 2) - 3\sin x(-2\cos x \sin x)$$
$$= -3\cos^3 x - 6\cos x + 6\cos x \sin^2 x = -9\cos^3 x.$$

Note that $f'(x) = 0$ when $\sin x = 0$, (since $\cos^2 x + 2 \geqslant 2$) that is, at $x = 0,\ \pi,\ 2\pi$. We have $f'(x) > 0$ on $(\pi, 2\pi)$ and $f'(x) < 0$ on $(0, \pi)$. Hence $x = \pi$ is a local minimum. So f increases on $[\pi, 2\pi]$ and decreases on $[0, \pi]$. Note that $f''(x) > 0$ on $(0, \pi/2)$ and on $(3\pi/2, 2\pi)$; $f''(x) < 0$ on $(\pi/2, 3\pi/2)$. So the function is concave up on $[0, \pi/2]$ and $[3\pi/2, 2\pi]$ and concave down on $[\pi/2, 3\pi/2]$. We have $f(0) = 7,\ f(\pi/2) = 0,\ f(\pi) = -7,\ f(3\pi/2) = 0,\ f(2\pi) = 7$. The graph is